N-POINT VIRASORO ALGEBRAS ARE MULTI-POINT KRICHEVER–NOVIKOV TYPE ALGEBRAS

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Abstract. We show how the recently again discussed N-point Witt, Virasoro, and affine Lie algebras are genus zero examples of the multi-point versions of Krichever–Novikov type algebras as introduced and studied by Schlichenmaier. Using this more general point of view, useful structural insights and an easier access to calculations can be obtained. The concept of almost-grading will yield information about triangular decompositions which are of importance in the theory of representations. As examples the algebra of functions, vector fields, differential operators, current algebras, affine Lie algebras, Lie superalgebras and their central extensions are studied. Very detailed calculations for the three-point case are given.

1. Introduction

Recently there was again a revived interest in algebras of meromorphic objects (vector fields, Lie algebra valued functions, and more) on the Riemann sphere [12], [14], [25], [30], [34]. In particular, this interest comes from representation theory and its interpretations in the context of quantization of (conformal) field theory. The appearing algebras supply examples of infinite dimensional (Lie) algebras which are of geometric origin. They generalize the Witt and Virasoro algebra respectively the classical affine Lie algebras. In some of these articles the vector field algebras were called N-Virasoro algebras. Here we like to stress the fact, that these algebras are nothing else as the genus zero Krichever–Novikov (KN) type algebras in their multi-point version as introduced by the current author.

Originally KN algebras were introduced by Krichever and Novikov in 1986/87 [31], [32], [33] with the intention to generalize the classical infinite-dimensional algebras of Conformal Field Theory (CFT) to higher genus. The classical algebras correspond to the geometric situation of genus zero and two fixed points where poles are allowed. In the original Krichever – Novikov approach still only two possible points for poles are considered. In the years 1989/1990 the author extended the whole set-up to the multi-point case for arbitrary genus (including genus zero) [44], [45], [46], [47].

In this work I will show that the consequent use of the techniques developed in these articles and the follow-ups [50], [51], adapted to the genus zero situation, will yield a much better understanding of the situation. It will explain certain properties remarked by the authors of [12], [14], [25], and remove some misconceptions.

The first nontrivial part in the multi-point extension is the fact that it is possible to assign to every non-empty splitting of the set A of points where poles are allowed an almost-graded structure

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(see Definition 4.1). This is done by choosing a basis adapted to the splitting. An almost-grading is very close to a grading. The notion is strong enough to construct triangular decompositions, semi-infinite wedge forms, Fock space representations, etc. These concepts are of relevance in representation theory and field theory. In the original Krichever–Novikov approach (and in the classical case) there is only one splitting possible. Hence, the additional effects due to different splittings do not show up. They were studied the first time in [46]. One of the consequences of the almost-grading will be that in the structure equations (with respect to the adapted basis) the number of basis elements appearing on the r.h.s. will be bounded by an uniform constant.

In the genus zero case with N points $P_1, P_2, \dots, P_N$ where poles are allowed, one could single out one point, e.g. P_N and move it by a fractional linear transformation to the point ∞. Having done this we could take

$$\{P_1, P_2, \dots, P_{N-1}\} \quad \cup \quad \{\infty\} \tag{1.1}$$

as one possible splitting. We will denote this splitting in this article *standard splitting*, despite the fact that it will depend on which point was chosen to be moved to ∞. Also we point out that for $N > 2$ this is just one example of a splitting. Now we are in the situation that we could employ all the constructions for the Krichever-Novikov type algebras carried out by the author. For the convenience of the reader will recall them for the genus zero case. Furthermore, we will strengthen the results if possible. No previous knowledge of KN type algebras will be needed. As a starting point we introduce the Poisson algebra of meromorphic forms. From this algebra the associative algebra of functions, the Lie algebra of vector fields and differential operators, superalgebras, current algebras, etc. are deduced.

Central extensions are of fundamental importance in the context of regularization of actions. Such regularizations are needed e.g. in quantum field theory. As our presented algebras are assumed to act as symmetry algebras, the classification of central extensions is a crucial task to be done. Of course, one is interested to classify all central extensions up to equivalence. But coming from the applications in representation theory one also needs to know which central extensions admit an extension of the almost-grading of the original algebra. Central extensions are defined via Lie algebras two-cocycles with values in the trivial module. A complete classification of those two-cycles classes which allow the extension of the almost-grading (those cocycles are called *local*, see Definition 5.1), and more general of the bounded ones for all the above Lie algebras are given by the author in [50], [51], [54]. This is done always with respect to a fixed splitting and induced almost-grading. The cocycles are given by a differential to be integrated about special integration paths, respectively by calculating residues.

One of the main results of the current article is that for genus zero each cocycle class is a bounded class with respect to the standard splitting (Theorem 5.11, Theorem 5.24, Theorem 5.31). For the function algebra we need to add the natural requirement for the cocycle to be $\mathcal{L}$-invariant or equivalently multiplicative. We use our earlier results about bounded cocycles to give them explicitly. Furthermore, we show in genus zero that the vector field algebra and the differential operator algebra are perfect. As we know their cocycle classes we can write down the universal central extension. The dimension of the center for the vector field algebra is $N-1$; for the differential operator algebra it is $3(N-1)$. The existence of an universal central extension for the vector field algebra for arbitrary genus is known [60], but here we supply an elementary proof. The algebra obtained as central extension from the function algebra via geometric cocycles (which might be called the *maximal Heisenberg algebra*), has also a $(N-1)$-dimensional center. The current algebra associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ admits also an universal central extension which has an $(N-1)$-dimensional center [51], [6]. For the vector field algebra, current algebra and function algebra there will be a unique (up to equivalence and rescaling) central extension compatible with the almost-grading, i.e. a local cocycle class. For the differential operator algebra the space of local cohomology classes will be three-dimensional.

After having developed the general picture for genus zero and N-points we show how to do explicit calculations in the 3-point case. From the side of applications this case is also of special interest, see e.g. [3], [23]. We normalize the three points to be $\{0, 1\} \cup \{\infty\}$ and use additional symmetry operations. All calculations reduce to calculations of residues of rational functions. This is also the case for general N.

In Section 8 we recall some types of representations which will be automatically available after having identified the algebras as special cases of KN type algebras.

In this article we mostly use the language of Riemann surfaces. But all definitions, objects, and results clearly make sense for arbitrary algebraically closed fields $\mathbb{K}$ of characteristics zero. Our Riemann sphere will be the projective line over $\mathbb{K}$.

It should be possible to study this article without consulting the works on KN type algebras mentioned above, assuming that the reader is willing to accept the statements of the theory. In case that the reader wants to know more we refer in addition to the original work also to the recent monograph [55] containing everything what is needed.

2. The Virasoro algebra and its relatives

The Virasoro algebra together with its relatives are the simplest non-trivial infinite dimensional Lie algebras. As we will generalize them we recall their definitions here.

2.1. **The Witt algebra.** The *Witt algebra* $\mathcal{W}$, sometimes also called Virasoro algebra without central term, is the Lie algebra generated as vector space by the basis elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m-n)\, e_{n+m}, \quad n, m \in \mathbb{Z}. \tag{2.1}$$

The algebra $\mathcal{W}$ is a graded Lie algebra. We define the degree by $\deg(e_n) := n$ and get the vector space direct decomposition

$$\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n, \qquad \mathcal{W}_n = \langle e_n \rangle_{\mathbb{C}}. \tag{2.2}$$

Obviously, $\deg([e_n, e_m]) = \deg(e_n) + \deg(e_m)$.

Algebraically $\mathcal{W}$ can also be given as Lie algebra of derivations of the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$.

2.2. **The Virasoro algebra.** For the Witt algebra the universal central extension is the *Virasoro algebra* $\mathcal{V}$. As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. We set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$. Its basis elements are $\hat{e}_n$, $n \in \mathbb{Z}$ and t with the Lie product [1].

$$[\hat{e}_n, \hat{e}_m] = (m-n)\, \hat{e}_{n+m} + \frac{1}{12}(n^3 - n)\delta_m^{-n}\, t, \qquad [\hat{e}_n, t] = [t, t] = 0, \tag{2.3}$$

for all $n, m \in \mathbb{Z}$. The factor $1/12$ is conventional. By setting $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ the Lie algebra $\mathcal{V}$ becomes a graded algebra. Up to equivalence of central extensions and rescaling the central element t, this is beside the trivial (splitting) central extension the only central extension of $\mathcal{W}$.

[1] Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero.

2.3. **The affine Lie algebra.** Given $\mathfrak{g}$ a finite-dimensional Lie algebra (e.g. a finite-dimensional simple Lie algebra) then the tensor product of $\mathfrak{g}$ with the associative algebra of Laurent polynomials $\mathbb{C}[z,z^{-1}]$ carries a Lie algebra structure via

$$[x\otimes z^n, y\otimes z^m] := [x,y]\otimes z^{n+m}, \quad x,y\in\mathfrak{g}\,. \tag{2.4}$$

This algebra is called *current algebra* or *loop algebra* and denoted by $\overline{\mathfrak{g}}$. Again we consider central extensions. For this let β be a symmetric, bilinear form for $\mathfrak{g}$ which is invariant (e.g. $\beta([x,y],z) = \beta(x,[y,z])$ for all $x,y,z\in\mathfrak{g}$). Then a central extension is given by

$$[\widehat{x\otimes z^n}, \widehat{y\otimes z^m}] := \widehat{[x,y]\otimes z^{n+m}} - \beta(x,y)\cdot n\,\delta_m^{-n}\cdot t. \tag{2.5}$$

This algebra is denoted by $\widehat{\mathfrak{g}}$ and called *affine Lie algebra*. With respect to the classification of Kac-Moody Lie algebras, in the case of a simple $\mathfrak{g}$ they are exactly the Kac-Moody algebras of untwisted affine type, [26], [27], [40].

2.4. **The geometric interpretation.** Let S^2 be the Riemann sphere or equivalently $\mathbb{P}^1(\mathbb{C})$ the projective line over $\mathbb{C}$. Denote by z the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$. The above algebras and basis elements have a geometric meaning. The elements are meromorphic objects which are holomorphic outside $\{0,\infty\}$. For example the algebra $\mathbb{C}[z,z^{-1}]$ can be given as the algebra of meromorphic functions on $S^2=\mathbb{P}^1(\mathbb{C})$ holomorphic outside of $\{0,\infty\}$.

The elements of the Witt algebra coincide with meromorphic vector fields, the element of the current algebra with $\mathfrak{g}$-valued meromorphic functions and so on. The Lie algebra structure of $\mathcal{W}$ corresponds to the usual Lie bracket of vector fields

$$[v,u] = \left(v\frac{d}{dz}u - u\frac{d}{dz}v\right)\frac{d}{dz}. \tag{2.6}$$

The basis elements are realized as $e_n = z^{n+1}\frac{d}{dz}$.

The central terms also have a geometric meaning which will become clear in the context of the generalization of these algebras described in the next sections.

3. The general N point and genus zero case

Even if in this article we are only interested in genus zero it is necessary to spend a few words on the arbitrary genus situation. In this context the full geometric meaning will become clearer. After we have done this we will concentrate on the genus zero case.

3.1. **Arbitrary genus and multi-point situation.** As explained in Section 2.4 in the geometric interpretation for the Virasoro algebra the objects are defined on the Riemann sphere and might have poles at most at two fixed points (which can be normalized to 0 and ∞) where poles are allowed. In applications, e.g. for a global operator approach to conformal field theory and its quantization, integrable systems, etc., this is not sufficient. One needs Riemann surfaces of arbitrary genus. Moreover, one needs more than two points where poles are allowed. Such generalizations were initiated by Krichever and Novikov [31], [32], [33], who considered arbitrary genus and the two-point case. This was extended to the multi-point case (and this case is of relevance here) and systematically examined by the author [44], [45], [46], [47], [48], [49], [50], [51], [55].

For the moment let Σ_g be a compact Riemann surface without any restriction for its genus $g=g(\Sigma_g)$. Furthermore, let A be a finite subset of Σ_g. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A=I\cup O$. Set $N:=\#A\ge 2$, $K:=\#I$, $M:=\#O$, with $N=K+M$. More precisely, let

$$I=(P_1,\ldots,P_K), \quad \text{and} \quad O=(Q_1,\ldots,Q_M) \tag{3.1}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i,j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes, we consider I and O simply as sets.

Our algebraic objects will be objects meromorphic on Σ_g and holomorphic outside of A. The corresponding algebras are called Krichever-Novikov type algebras. An almost-grading (see Definition 4.1) is introduced with respect to the splitting of A. The general theory can be found in the above references and will not be repeated here. For the rest of this contribution we will specialize the theory for the genus zero multi-point situation. Of course, for this the multi-point theory developed by the author plays an important role.

3.2. **Genus zero.** Now let Σ_0 be the Riemann sphere S^2, or equivalently $\mathbb{P}^1(\mathbb{C})$ with the quasi-global coordinate z. We call it quasi-global, as it is not defined at the point ∞. Let us denote the set of points

$$A = \{P_1, P_2, \dots, P_N\}, \quad P_i \neq P_j, \text{ for } i \neq j. \tag{3.2}$$

For notational simplicity we single out the point P_N as reference point. By an automorphism of $\mathbb{P}^1(\mathbb{C})$, i.e. a fractional linear transformation or equivalently an element of $\mathrm{PGL}(2,\mathbb{C})$, the point P_N can be brought to ∞. In fact two more points could be normalized to be 0 and 1. In this section we will not do so, but see Section 6.

Our points are given by their coordinates

$$P_i = a_i, \quad a_i \in \mathbb{C}, \ i = 1, \dots, N-1, \qquad P_N = \infty. \tag{3.3}$$

At these points we have the local coordinates

$$z - a_i, \ i = 1, \dots, N-1, \qquad w = 1/z. \tag{3.4}$$

Sometimes we refer to the classical situation. By this we understand

$$\Sigma_0 = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{0\}, \quad O = \{\infty\}. \tag{3.5}$$

3.3. **Meromorphic forms.** To introduce the elements of the algebras we first have to consider forms of (conformal) weight $\lambda \in 1/2\mathbb{Z}$. Without further saying, we always assume that they are meromorphic and holomorphic outside of A.

Forms of weight 0 are functions. Of course they constitute an associative algebra, which we denote by $\mathcal{A}$. Forms of weight 1 are (meromorphic) differentials. Recall that the canonical line bundle $\mathcal{K}$ of Σ is the holomorphic line bundle whose local sections are the the local holomorphic differentials. For $\mathbb{P}^1(\mathbb{C})$, in the language of algebraic geometry, we have that $\mathcal{K} = \mathcal{O}(-2)$. This bundle has a unique square root $L = \mathcal{O}(-1)$, [2] which is the tautological bundle, respectively the dual of the hyperplane section bundle. We denote this bundle also by $\mathcal{K}^{1/2}$.

Meromorphic forms of weight λ are sections of the bundle $\mathcal{K}^\lambda$ where (with $\mathcal{K}^*$ the dual bundle of $\mathcal{K}$) (1) $\mathcal{K}^\lambda = \mathcal{K}^{\otimes\lambda}$ for $\lambda > 0$, (2) $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle, and (3) $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$. We set

$$\mathcal{F}^\lambda := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid f \text{ is holomorphic on } \Sigma \setminus A\}. \tag{3.6}$$

Obviously this is an infinite dimensional $\mathbb{C}$-vector space. Its elements are called *meromorphic forms of weight* λ. In particular, $\mathcal{F}^0 = \mathcal{A}$. Forms of weight -1 are (meromorphic) vector fields and we set $\mathcal{L} := \mathcal{F}^{-1}$.

In local coordinates z_i a form of weight λ can be written as $b_i(z)(dz_i)^\lambda$ with b_i a meromorphic function. By $(dz_i)^\lambda$ it is encoded how the local functions transform under coordinate transformation. In our genus zero situation this simplifies. We can describe the form by a meromorphic

[2] This is not true anymore for Riemann surfaces of higher genera g. In fact, we have 2^{2g} different square roots. They correspond to different spin structures.

function on the affine part $\mathbb{C}$ with respect to the coordinate z. By this description its behaviour at the point ∞ is uniquely fixed by the fundamental transformation $dz = -w^{-2}dw$. Moreover, by the fixing $P_N = \infty$ the set of meromorphic forms f of weight λ on $\mathbb{P}^1(\mathbb{C})$ holomorphic outside of A correspond 1:1 to meromorphic functions $a(z)$ holomorphic outside of A via $f(z) = a(z)dz^\lambda$. Both a and f will have the same orders at the points in $\mathbb{C}$. For the order at the point ∞ we have

$$\operatorname{ord}_\infty(f) = \operatorname{ord}_\infty(a) - 2\lambda. \tag{3.7}$$

Recall that on a compact Riemann surface the sum of the orders (summed over all points) of a meromorphic function $f \not\equiv 0$ equals zero. Hence, more generally

Proposition 3.1. *Let $f \in \mathcal{F}^\lambda$, $f \not\equiv 0$ then*

$$\sum_{P \in \Sigma_0} \operatorname{ord}_P(f) = -2\lambda. \tag{3.8}$$

For this and related results see e.g. [52]. Also recall that the meromorphic functions in our case are nothing else as rational functions with respect to the variable z.

Next we introduce algebraic operations on the vector space of meromorphic forms of arbitrary weights (integer or half-integer). We introduce the space

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda, \tag{3.9}$$

obtained by summing over all weights. The basic operations will allow us to introduce finally the algebras we are heading for. The following constructions make perfect sense in the arbitrary genus case and the statement have been proven there. For completeness we recall the results.

3.4. **Associative structure.** The natural map of the locally free sheaves of rang one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{3.10}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{3.11}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto s\,dz^\lambda \cdot t\,dz^\nu = s \cdot t\; dz^{\lambda+\nu}. \tag{3.12}$$

Proposition 3.2. *The space $\mathcal{F}$ is an associative and commutative graded (over $\frac{1}{2}\mathbb{Z}$) algebra. Moreover, $\mathcal{A} = \mathcal{F}^0$ is a subalgebra and the $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.*

3.5. **Lie and Poisson algebra structure.** There is a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (e,f) \mapsto [e,f], \tag{3.13}$$

which is defined in local representatives of the sections by

$$(e\,dz^\lambda, f\,dz^\nu) \mapsto [e\,dz^\lambda, f\,dz^\nu] := \left((-\lambda)e\frac{df}{dz} + \nu f\frac{de}{dz}\right) dz^{\lambda+\nu+1}, \tag{3.14}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 3.3. [55, Prop. 2.6 and 2.7] *The prescription $[.,.]$ given by (3.14) is well-defined and defines a Lie algebra structure on the vector space $\mathcal{F}$.*

Proposition 3.4. [55, Prop. 2.8] *The subspace $\mathcal{L} = \mathcal{F}^{-1}$ is a Lie subalgebra, and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.*

Theorem 3.5. [55, Thm. 2.10] *The triple* $(\mathcal{F}\ ,\ \cdot\ ,\ [.,.])$ *is a Poisson algebra.*

As substructures we already encountered the subalgebras $\mathcal{A}$ of meromorphic functions and the subalgebra $\mathcal{L}$ of meromorphic vector fields. The spaces $\mathcal{F}^\lambda$ are modules over them.

For the vector fields we obtain by the above the usual Lie bracket and the usual Lie derivative for their actions on forms. Written explicitly for the vector fields we get

$$[e,f]_| = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ , \tag{3.15}$$

for $e, f \in \mathcal{L}$. We used the same symbol for the vector field and for the representing function. For the Lie derivative we get

$$\nabla_e(f)_| = L_e(g)_| = e\,.\,g_| = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{3.16}$$

3.6. **The algebra of differential operators.** The Lie algebra $\mathcal{F}$, has $\mathcal{F}^0$ as an abelian Lie subalgebra. The vector space sum $\mathcal{D}^1 = \mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L}$ is also a Lie subalgebra. In an equivalent way this can also be constructed as semidirect sum of $\mathcal{A}$ considered as abelian Lie algebra and $\mathcal{L}$ operating on $\mathcal{A}$ by taking the derivative. The algebra $\mathcal{D}^1$ is the *Lie algebra of differential operators of degree* ≤ 1. In terms of elements the Lie product is

$$[(g,e),(h,f)] = (e\,.\,h - f\,.\,g\ ,\ [e,f]). \tag{3.17}$$

The projection on the second factor $(g,e) \mapsto e$ is a Lie homomorphism and we obtain a short exact sequences of Lie algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{D}^1 \longrightarrow \mathcal{L} \longrightarrow 0\ . \tag{3.18}$$

Hence $\mathcal{A}$ is an (abelian) Lie ideal of $\mathcal{D}^1$ and $\mathcal{L}$ a quotient Lie algebra. Obviously, $\mathcal{L}$ is also a subalgebra of $\mathcal{D}^1$.

The vector space $\mathcal{F}^\lambda$ becomes a Lie module over $\mathcal{D}^1$ by the operation

$$(g,e).f := g\cdot f + e.f, \quad (g,e) \in \mathcal{D}^1(A), f \in \mathcal{F}^\lambda(A). \tag{3.19}$$

Via some universal constructions differential operator algebras of arbitrary degrees can be constructed. We will not repeat their definition here, but only refer to [55, Chap. 2.7]

3.7. **Current algebras.** We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$. The generalized current algebra is defined as $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x,y] \otimes f \cdot g, \qquad x, y \in \mathfrak{g}, \quad f, g \in \mathcal{A}. \tag{3.20}$$

It can be easily verified that $\overline{\mathfrak{g}}$ is a Lie algebra.

Later we will introduce central extensions for these current algebras. They will generalize affine Lie algebras, respectively affine Kac-Moody algebras of untwisted type.

3.8. **$\mathfrak{g}$-differential operators.** For some applications (e.g. for the fermionic Fock space representations, for the Sugawara representation) it is useful to extend the definition of the current algebras by considering differential operators (of degree ≤ 1) associated to $\overline{\mathfrak{g}}$. We define $\mathcal{D}^1_{\mathfrak{g}} := \overline{\mathfrak{g}} \oplus \mathcal{L}$ and take in the summands the Lie products defined there and put additionally

$$[e, x \otimes g] := -[x \otimes g, e] := x \otimes (e.g). \tag{3.21}$$

This operation can be described as semidirect sum of $\overline{\mathfrak{g}}$ with $\mathcal{L}$ and we get

Proposition 3.6. [55, Prop. 2.15] $\mathcal{D}^1_{\mathfrak{g}}$ *is a Lie algebra.*

All the above algebras I call Krichever–Novikov type algebras, despite the fact that Krichever and Novikov did not introduce all these types of algebras. Furthermore, they only considered the two-point case.

4. Choice of a basis and an almost-grading

4.1. **Definition of an almost-grading.** In the classical situation the introduced algebras are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [31] there is a weaker concept, an almost-grading, which to a large extent is a valuable replacement of a honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting a certain basis of the spaces $\mathcal{F}^\lambda$ and define these elements to be (almost-) homogeneous.

Definition 4.1. Let $\mathcal{L}$ be a Lie or an associative algebra such that

$$\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n \tag{4.1}$$

is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

- (i) $\dim \mathcal{L}_n < \infty$,
- (ii) There exists constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

If $\dim \mathcal{L}_n$ is bounded with a bound independent of n we call $\mathcal{L}$ *strongly almost-graded*. If we drop the condition that $\dim \mathcal{L}_n$ is finite we call $\mathcal{L}$ *weakly almost-graded*.

In a similar manner almost-graded modules over almost-graded algebras are defined. Furthermore, this definition makes complete sense also for more general index sets $\mathbb{J}$.

4.2. **Existence of an almost-grading.** Given a splitting $A = I \cup O$, one of the results of the author is that all the above introduced algebras $\mathcal{A}, \mathcal{L}, \mathcal{D}^1$, $\overline{\mathfrak{g}}$, and $\mathcal{D}^1_{\mathfrak{g}}$ admit a strongly almost-graded structure induced by a well-defined procedure given by fixing an adapted basis. Essentially different splittings (meaning not just obtained by inverting the role of I and O) will yield “non-equivalent” almost-gradings.

Remark 4.2. In the two-point case, i.e. the classical case, or more generally the case considered by Krichever and Novikov, the set A consists of two points. Hence, there is only one splitting possible and consequently only one almost-grading. This is not the case anymore for more than two points. To realize the importance of the splitting was a crucial observation by the author [47], [46].

We set $\mathbb{J}_\lambda = \mathbb{Z}$ for $\lambda \in \mathbb{Z}$ and $\mathbb{J}_\lambda = \mathbb{Z} + 1/2$ for $\lambda \in \mathbb{Z} + 1/2$. Given a splitting with $\#I = K$ the author gives a procedure to exhibit a certain basis of $\mathcal{F}^\lambda$

$$\{f^\lambda_{n,p} \mid n \in \mathbb{J}_\lambda,\ p = 1, \ldots, K\} \tag{4.2}$$

with special properties. The subspace

$$\mathcal{F}^\lambda_n := \langle f^\lambda_{n,p} \mid p = 1, \ldots, K\rangle_{\mathbb{C}} \quad \subset\ \mathcal{F}^\lambda \qquad n \in \mathbb{J}_\lambda \tag{4.3}$$

is the subspace of homogeneous elements of degree n. Then

$$\mathcal{F}^\lambda = \bigoplus_{n\in\mathbb{J}_\lambda} \mathcal{F}^\lambda_n, \quad \dim \mathcal{F}^\lambda_n = K. \tag{4.4}$$

As we will give in the following an explicit construction we will not need details of the general theory. We refer the interested reader e.g. to [44], [46], [55] for details.

Let us numerate in our genus zero situation the points in the splitting like

$$A = I \cup O, \qquad I = (P_1, P_2, \ldots, P_K), \quad O = (P_{K+1}, \ldots, P_N = \infty). \tag{4.5}$$

As $P_N = \infty \in O$ it is enough to construct a basis $\{A_{n,p} \mid n \in \mathbb{Z}, p = 1, \ldots K\}$ for $\mathcal{A} = \mathcal{F}^0$. The decomposition of $\mathcal{A}$ induces a decomposition of $\mathcal{F}^\lambda$ by

$$\mathcal{F}^\lambda_n = \mathcal{A}_{n-\lambda} dz^\lambda, \quad \text{respectively} \quad f^\lambda_{n,p} = A_{n-\lambda,p} dz^\lambda. \tag{4.6}$$

The shift by $-\lambda$ is quite convenient and is beside other things related to the duality property discussed further down. The recipe for constructing the $A_{n,p}$ is given in [46], [45], see also [55]. As a principal property we have

$$\operatorname{ord}_{P_i}(A_{n,p}) = (n+1) - \delta_i^p, \; i = 1, \ldots, K\,. \tag{4.7}$$

At the points in O corresponding orders are set to make the element unique up to multiplication by a non-zero scalar.

Example. We call the splitting

$$I = (P_1, P_2, \ldots, P_K), \quad O = (P_N = \infty), \quad K = N - 1 \tag{4.8}$$

the *standard splitting*. Recall that P_i corresponds to the point given by the coordinate $z = a_i$. We set

$$\alpha(p) := \Big(\prod_{\substack{i=1\\ i\neq p}}^K (a_p - a_i) \Big)^{-1}. \tag{4.9}$$

and define

$$A_{n,p}(z) := (z - a_p)^n \cdot \prod_{\substack{i=1\\ i\neq p}}^K (z - a_i)^{n+1} \cdot \alpha(p)^{n+1}\,. \tag{4.10}$$

The last factor is a normalization factor yielding

$$A_{n,p}(z) = (z - a_p)^n (1 + O(z - a_p)). \tag{4.11}$$

With this description the order at ∞ is fixed as

$$-(Kn + K - 1)\,. \tag{4.12}$$

By (4.6) the basis elements for the other $\mathcal{F}^\lambda$ are given too. In particular, we obtain for $\mathcal{L}$ the basis

$$e_{n,p} = f^{-1}_{n,p} = A_{n+1,p} \frac{d}{dz} = (z - a_p)^{n+1} \cdot \prod_{\substack{i=1\\ i\neq p}}^K (z - a_i)^{n+2} \cdot \alpha(p)^{n+2} \, \frac{d}{dz}. \tag{4.13}$$

4.3. **Duality.** The pairing which we describe here is valid for arbitrary genus and arbitrary splittings. Let C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, K$ and C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$. A cycle C_S is called a *separating cycle* if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points. It might have multiple components. In the following we will integrate meromorphic differentials on Σ_g without poles in $\Sigma_g \setminus A$ over closed curves C. Hence, we might consider the C and C' as equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{Z})$. In this sense we can write for every separating cycle C_S

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{4.14}$$

The minus sign appears due to the opposite orientation. Another way for giving such a C_S is via level lines of a "proper time evolution", for which I refer to [55, Section 3.9].

Given such a separating cycle C_S (respectively cycle class) we define a linear map

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega. \tag{4.15}$$

The map will not depend on the separating line C_S chosen, as two of such will be homologous and the poles of ω are only located in I and O.

Consequently, the integration of ω over C_S can also be described over the special cycles C_i or equivalently over C_j^*. This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \;=\; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega). \tag{4.16}$$

The pairing

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \to \mathbb{C}, \quad (f, g) \mapsto \langle f, g \rangle := \frac{1}{2\pi \mathrm{i}} \int_{C_S} f \cdot g, \tag{4.17}$$

between λ and $1 - \lambda$ forms is called *Krichever-Novikov (KN) pairing.*

With respect to this pairing we have [55, Thm. 3.6]

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{-m,r} \rangle = \delta_p^r \, \delta_n^m, \qquad \forall n, m \in \mathbb{J}_\lambda, \quad r, p = 1, \ldots, K. \tag{4.18}$$

In particular, the pairing is non-degenerate. This pairing is extremely helpful as e.g. given $f \in \mathcal{F}^\lambda$ then the expansion in terms of the basis

$$f = \sum_{n \in \mathbb{J}_\lambda} \sum_{p=1}^{K} \alpha^\lambda_{n,p} f^\lambda_{n,p}, \qquad \alpha_{n,p} \in \mathbb{C} \tag{4.19}$$

can be determined via

$$\alpha_{n,p} = \langle f, f^{1-\lambda}_{-n,p} \rangle = \sum_{P \in I} \mathrm{res}_P(f \cdot f^{1-\lambda}_{-n,p}) = -\sum_{Q \in O} \mathrm{res}_Q(f \cdot f^{1-\lambda}_{-n,p}). \tag{4.20}$$

Note that the pairing depends not only on A (as the $\mathcal{F}^\lambda$ depend on it) but also critically on the splitting of A into I and O as the integration path C_S will depend on it. Once the splitting is fixed the pairing will be fixed too.

4.4. **Almost-graded structure of the algebras.** From the general theory it follows that all our algebras $\mathcal{A}, \mathcal{L}, \mathcal{D}^1, \overline{\mathfrak{g}}, \mathcal{D}^1_{\mathfrak{g}}$ are strongly almost-graded and the $\mathcal{F}^\lambda$ are almost-graded modules over the first three. More precisely, there exists R_1 and R_2 such that

$$\begin{aligned} \mathcal{A}_n \cdot \mathcal{A}_m &\subseteq \bigoplus_{k=n+m}^{n+m+R_1} \mathcal{A}_k \\ [\mathcal{L}_n, \mathcal{L}_m] &\subseteq \bigoplus_{k=n+m}^{n+m+R_2} \mathcal{L}_k. \end{aligned} \tag{4.21}$$

Similar expressions are there for the modules with exactly the same bounds. The lowest order terms can be given as

$$\begin{aligned} A_{n,p} \cdot A_{m,r} &= A_{n+m,r}\,\delta_r^p \ + \ \text{h.d.t.} \\ A_{n,p} \cdot f^\lambda_{m,r} &= f^\lambda_{n+m,r}\,\delta_r^p \ + \ \text{h.d.t.} \\ [e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\,\delta_r^p \ + \ \text{h.d.t.} \\ e_{n,p} \cdot f^\lambda_{m,r} &= (m+\lambda n) \ \cdot f^\lambda_{n+m,r}\,\delta_r^p \ + \ \text{h.d.t.} \end{aligned} \tag{4.22}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_i$,

The other coefficients can be explicitly calculated. Also it is easy to determine the upper bounds R_1 and R_2. This is done by calculating the algebraic result of the basis elements and then expanding the result via (4.19) and calculate by (4.20) the coefficients via residues of rational functions. As an example we consider $\mathcal{A}$. The structure equations of $\mathcal{A}$ are given via

$$A_{n,p} \cdot A_{m,r} = \sum_{h \in \mathbb{Z}} \sum_{s=1}^{K} \alpha^{(h,s)}_{(n,p)(m,s)}\, A_{h,s}. \tag{4.23}$$

Recalling that $f^1_{-h,s} = A_{(-h-1,s)} dz$ we set

$$\omega = A_{n,p} \cdot A_{m,r} \cdot A_{(-h-1,s)} dz. \tag{4.24}$$

Then

$$\alpha^{(h,s)}_{(n,p)(m,s)} = \sum_{P \in I} \operatorname{res}_P(\omega) = -\sum_{Q \in O} \operatorname{res}_Q(\omega). \tag{4.25}$$

It is convenient to set $h = n+m+k$. By summing up all the orders of the factors of ω individually at the point $P \in I$ we see that there is no residue at I if $k < 0$. Hence the coefficients are vanishing in the sum (4.23) for $h < n+m$. Doing the same for the orders at the points Q in O we see that there is a bound R_1 such that if $k > R_1$ there will be no residue at O. Hence we get the almost-grading. The coefficients can be explicitly calculated by calculating residues of rational functions. The lowest term will only show up if $p = r = s$ and there is only a residue at P_p which is equal to 1.

Exactly the same kind of arguments work for the algebra $\mathcal{L}$ where we now obtain the bound R_2. The same is true for the modules $\mathcal{F}^\lambda$.

4.5. **Triangular decomposition.** Let $\mathcal{U}$ be one of the above introduced algebras (including the current algebra). On the basis of the almost-grading we obtain a triangular decomposition of the algebras

$$\mathcal{U} = \mathcal{U}_{[+]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[-]}, \tag{4.26}$$

where

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{4.27}$$

By the almost-gradedness the $[+]$ and $[-]$ subspaces are (infinite dimensional) subalgebras. The $\mathcal{U}_{[0]}$ are only finite-dimensional vector spaces.

Depending on the applications it is sometimes useful to enlarge the algebra $\mathcal{U}_{[-]}$ by adding a finite-dimensional subspace from $\mathcal{U}_{[0]}$ so that the enlarged algebra $\mathcal{U}^*_{[-]}$ contains all objects regular at the points in O, respectively vanishing there with a certain order.

The existence of such a triangular decomposition indicates the importance of the existence of an almost-grading. Such a triangular decomposition is necessary for developing a rich representation theory. On its basis one constructs highest weight representations, Verma modules, Fock space representations and many more. The elements of $\mathcal{U}_{[+]}$ quite often correspond to annihilation operators, the others to creation operators.

4.6. **Standard splitting.** For the standard splitting the set O consists only of the point ∞. The elements $A_{n,p}$ for $p = 1, \ldots, K = N-1$ are introduced above. For illustration we give the bounds R_1 and R_2

Proposition 4.3.

$$R_1 = \begin{cases} 0, & N = 2, \\ 1, & N > 2, \end{cases} \qquad\qquad R_2 = \begin{cases} 0, & N = 2, \\ 1, & N = 3, \\ 2, & N > 3\,. \end{cases} \tag{4.28}$$

Proof. For calculating the order at ∞ with respect to the variable w we use for the individual factors in the expression (4.24) the value (4.12) and sum over all factors and do not forget to decrease the order by 2 coming from dz. If we do this for the algebra $\mathcal{A}$ we get as order for $A_{n,p} \cdot A_{m,r} \cdot A_{-(n+m+k)-1}$ the value $-2K + K \cdot k + 1$. A pole is only possible if this value is ≤ -1. Hence only for

$$k \leq -\frac{2}{K} + 2\,. \tag{4.29}$$

This yields the claimed value for R_1. For the Lie algebra $\mathcal{L}$, respectively for the Lie module we have to consider $A_{n+1,p} \cdot A_{m-\lambda,r} \cdot A_{-(n+m+k)-(1-\lambda)}$. For the order at ∞ we obtain $-3K + K \cdot k + 2$. Which yields that a pole is only possible for

$$k \leq -\frac{3}{K} + 3\,, \tag{4.30}$$

and hence the claimed value for R_2. $\square$

The structure coefficients of the algebras can be directly calculated by calculating residues of rational functions via (4.20). We will not do it here for the general case. In Section 6 we treat the three-point case in detail.

Example. $N = 2$**.** By a $\mathrm{PGL}(2, \mathbb{C})$ action the two points can be transported to 0 and ∞. This is the classical situation and there is only one splitting. Hence, everything is fixed. The above basis gives back the conventional one.

Example. $N = 3$**.** Here by a $\mathrm{PGL}(2, \mathbb{C})$ action the three points can be normalized to $\{0, 1, \infty\}$. Hence, up to isomorphy there are only one three-point algebra (for each type). If we fix such an algebra we obtain three different splittings of the set $\{0, 1, \infty\}$ and consequently also 3 essentially different almost-gradings, triangular decompositions, etc. The three-point case is in a certain sense special as there are still the automorphism of $\mathbb{P}^1(\mathbb{C})$ permuting these three points. They induce automorphisms of the algebras which permute the almost-gradings. We will consider this situation in detail in Section 6.

Example. $N = 4$**.** This is the first case where we have a moduli parameter for the geometric situation. We normalize our A to

$$\{0, 1, a, \infty\}, \qquad a \in \mathbb{C},\ a \neq 0, 1\,. \tag{4.31}$$

We have 2 different types of splittings, i.e. the type $4 = 3+1$ and the type $4 = 2+2$. For example

$$\{0, 1, a\} \cup \{\infty\}, \quad \text{and} \quad \{0, 1\} \cup \{a, \infty\}\,. \tag{4.32}$$

The first type is the standard splitting for which we gave the basis above. For the second splitting a basis of $\mathcal{A}$ and hence of all $\mathcal{F}^\lambda$ is

$$\begin{aligned} A_{n,1}(z) &= z^n (z-1)^{n+1} (z-a)^{-(n+1)} a^{(n+1)}, \\ A_{n,2}(z) &= z^{n+1} (z-1)^n (z-a)^{-(n+1)} (1-a)^{(n+1)}\,, \end{aligned} \tag{4.33}$$

where $n \in \mathbb{Z}$. The last factor is again a normalization constant. This basis defines an almost-grading for the four-point algebra which is not equivalent to the standard almost-grading. Again upper bounds for the level of the algebra coefficients and the coefficients itself can be calculated easily via residues.

4.7. **Another basis.** Clearly, our algebra $\mathcal{A}$ can be given as the algebra

$$\mathcal{A} = \mathbb{C}[(z-a_1), (z-a_1)^{-1}, (z-a_2)^{-1}, \ldots, (z-a_{N-1})^{-1}], \tag{4.34}$$

with the obvious relations.

If we introduce

$$A_n^{(i)} := (z-a_i)^n, \tag{4.35}$$

then

$$A_n^{(i)}, \quad n \in \mathbb{Z},\ i = 1, \ldots, N-1 \tag{4.36}$$

is a generating set of $\mathcal{A}$. A basis is given e.g. by

$$A_n^{(1)},\ n \in \mathbb{Z}, \quad A_{-n}^{(i)},\ n \in \mathbb{N},\ i = 2, \ldots, N-1. \tag{4.37}$$

Lemma 4.4. *Let $\mathcal{A}_{(0)}$ be the subalgebra of meromorphic functions holomorphic outside of ∞ then*

$$\mathcal{A}_{(0)} = \langle A_n^{(1)},\ n \geq 0 \rangle_{\mathbb{C}} = \langle A_n^{(2)},\ n \geq 0 \rangle_{\mathbb{C}} = \cdots = \langle A_n^{(N-1)},\ n \geq 0 \rangle. \tag{4.38}$$

Explicit calculations with respect to the basis (4.37) have been done in [44], [47]. Similar calculations were done e.g. by Dick [15], Anzaldo-Meneses [1] and Guo, Na, Shen, Wang, Yu [21]. It turns out that for the "products" of certain type of elements of this basis the number of elements in the results do not have a bound. Hence, it is not possible to introduce a strong almost-grading of $\mathcal{A}$ such that these basis elements are homogeneous and it is not possible to construct triangular decompositions with respect to this basis. After realizing this I switched in [46] to the kind of basis presented above. Nevertheless this kind of basis will play a role in some proofs later. Of course, as above, a basis of $\mathcal{A}$ will yield a basis of $\mathcal{L}$ (and $\mathcal{F}^\lambda$). We will also use

$$e_n^{(i)} = A_{n+1}^{(i)} \frac{d}{dz}, \qquad n \in \mathbb{Z},\ i = 1, \ldots, N-1. \tag{4.39}$$

5. Central extensions

Central extension of our algebras appear naturally in the context of quantization and the regularization of actions. Of course, they are also of independent mathematical interest.

5.1. **Central extensions and cocycles.** For the convenience of the reader let us repeat the relation between central extensions and the second Lie algebra cohomology with values in the trivial module. A central extension of a Lie algebra $\mathcal{U}$ is a special Lie algebra structure on the vector space direct sum $\widehat{\mathcal{U}} = \mathbb{C} \oplus \mathcal{U}$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ then the Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \psi(x, y) \cdot t, \quad [t, \widehat{\mathcal{U}}] = 0, \quad x, y \in \mathcal{U}\,, \tag{5.1}$$

with bilinear form ψ. The map $x \mapsto \hat{x} = (0, x)$ is a linear splitting map. $\widehat{\mathcal{U}}$ will be a Lie algebra, e.g. will fulfill the Jacobi identity, if and only if ψ is an antisymmetric bilinear form and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\psi(x, y, z) := \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y). \tag{5.2}$$

A 2-cochain ψ is a coboundary if there exists a linear form $\varphi : \mathcal{U} \to \mathbb{C}$ such that

$$\psi(x, y) = \varphi([x, y]). \tag{5.3}$$

Every coboundary is a cocycle. The second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U}, \mathbb{C})$ of $\mathcal{U}$ with values in the trivial module $\mathbb{C}$ is defined as the quotient of the space of 2-cocycles modulo coboundaries.

Two central extensions are equivalent if they essentially differ by the choice of the splitting maps. They are equivalent if and only if the difference of their defining 2-cocycles ψ and ψ' is a coboundary. In this way the second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U}, \mathbb{C})$ classifies equivalence classes of central extensions. The class [0] corresponds to the trivial central extension. In this case the splitting map is a Lie homomorphism. To construct central extensions of our algebras we have to find such Lie algebra 2-cocycles.

Clearly, equivalent central extensions are isomorphic. The opposite is not true. In our case we can always rescale the central element by multiplying it with a nonzero scalar. This is an isomorphism but not an equivalence of central extensions. Nevertheless, it is an irrelevant modification. Hence we will be mainly interested in central extensions modulo equivalence and rescaling. They are classified by [0] and the elements of the projectivized cohomology space $\mathbb{P}(\mathrm{H}^2(\mathcal{U}, \mathbb{C}))$.

Recall that if $\mathcal{U}$ is a perfect Lie algebra, i.e. if $[\mathcal{U}, \mathcal{U}] = \mathcal{U}$ then there exists a universal central extension and $k = \dim \mathrm{H}^2(\mathcal{U}, \mathbb{C})$ gives the dimension of the center of this extension. Moreover, if this space is finite-dimensional then the universal central extension is up to equivalence given as follows. As vector space it is the direct sum $\widehat{\mathcal{U}}_{univ} = \mathbb{C}^k \oplus \mathcal{U}$. Let $[\psi_i]$, $i = 1, \ldots, k$ be a basis of $\mathrm{H}^2(\mathcal{U}, \mathbb{C})$ and each class represented by a cocycle ψ_i. Moreover, let $t_1, t_2, \ldots, t_k$ be standard basis elements of $\mathbb{C}^k$, then the Lie structure is given as

$$[\widehat{x}, \widehat{y}] = \widehat{[x, y]} + \sum_{i=1}^{k} \alpha_i \psi_i(x, y) \cdot t_i, \quad x, y \in \mathcal{U},\ \alpha_i \in \mathbb{C} \quad [t_i, \widehat{\mathcal{U}}_{univ}] = 0\,. \tag{5.4}$$

5.2. **Almost-graded central extensions.** Before we discuss for each of our algebras the central extensions separately we will treat their common features. Denote by $\mathcal{U}$ one of these algebras. Our algebras are almost-graded. Coming from the applications one is quite often only interested in central extensions $\widehat{\mathcal{U}}$ which allow to extend the almost-grading of $\mathcal{U}$. This says that only those cocycles are allowed such that it is possible to assign to the central element t a fixed degree such that

$$[\widehat{\mathcal{U}}_n, \widehat{\mathcal{U}}_m] \quad \subseteq \quad \sum_{h=n+m-L_1}^{n+m+L_2} \widehat{\mathcal{U}}_h, \tag{5.5}$$

with L_1 and L_2 independent of n and m. If there is such a value for the degree of t, the value $\deg t = 0$ will also do. Hence, without restriction we will take this value. For $\hat{x} = (x, 0)$ and

$t = (0, 1)$ we set

$$\deg \hat{x} = \deg x, \quad \deg t = 0. \tag{5.6}$$

Definition 5.1. (a) Let γ be a 2-cocycle for the almost-graded Lie algebra $\mathcal{U}$, then γ is called a *local cocycle* if $\exists T_1, T_2$ such that

$$\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies T_2 \leq n + m \leq T_1. \tag{5.7}$$

(b) A 2-cocycle γ is called *bounded* (from above) if $\exists T_1$ such that

$$\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies n + m \leq T_1. \tag{5.8}$$

(c) A cocycle class $[\gamma]$ is called a *local (bounded) cohomology class* if and only if it admits a representing cocycle which is local (respectively bounded).

Note that e.g. in a local cocycle class not all representing cocycles are local. Obviously, the set of local (or bounded) cocycles is a subspace of all cocycles. Moreover, the set $\mathrm{H}^2_{loc}(\mathcal{U}, \mathbb{C})$ (respectively $\mathrm{H}^2_b(\mathcal{U}, \mathbb{C})$) of local (respectively bounded) cohomology classes is a subspace of the full cohomology space.

We point out that what is local (and bounded) depends on the almost-grading induced by the splitting $A = I \cup O$.

Remark 5.2. We could also introduce the space of $\widetilde{\mathrm{H}}^2_{loc}(\mathcal{U}, \mathbb{C})$ of local cocycles modulo local coboundaries. This space can naturally be identified with $\mathrm{H}^2_{loc}(\mathcal{U}, \mathbb{C})$ as if two local cocycles γ_1 and γ_2 are cohomologous then the corresponding coboundary is local too.

Remark 5.3. In the classical Witt algebra case $\mathrm{H}^2(\mathcal{W}, \mathbb{C}) = \mathrm{H}^2_{loc}(\mathcal{W}, \mathbb{C})$. The corresponding result is neither true for higher genus, nor for the multi-point situation.

As explained above the almost-grading is crucial for the triangular decomposition. Hence it should not be a surprise if the cocycles obtained via regularization processes from the usual representations are local, see [47], [55].

5.3. **Geometric cocycles.** Our algebras $\mathcal{U}$ consists of geometric objects. Hence, it is quite natural to have a closer look at Lie algebra cocycles which can be defined via geometric means.

Definition 5.4. A cocycle $\gamma : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ is called a *geometric cocycle* if there is a bilinear map

$$\widehat{\gamma} : \mathcal{U} \times \mathcal{U} \to \mathcal{F}^1, \tag{5.9}$$

such that γ is the composition of $\widehat{\gamma}$ with an integration, i.e.

$$\gamma = \gamma_C := \frac{1}{2\pi \mathrm{i}} \int_C \widehat{\gamma} \tag{5.10}$$

with C a curve on Σ_g. A cohomology class $[\gamma]$ is a *geometric cohomology class* if it contains a representing element $\gamma' \in [\gamma]$ which is a geometric cocycle.

Proposition 5.5. *For a geometric cocycle $\gamma_C = \gamma_{C'}$ if $[C] = [C'] \in \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$.*

Proof. As $\mathcal{F}^1$ is the space of meromorphic differentials holomorphic on $\Sigma_g \setminus A$ the integral over two cohomologous cycles will be the same. □

On this basis we can write $\gamma_{[C]}$ and even allow that $[C]$ is an arbitrary element from $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$.

In the opposite direction after fixing a bilinear map $\widehat{\gamma}$ the space $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ will parameterize (in a possibly non-unique way) the space of associated cocycles $\gamma_{[C]}$. This is exactly the way which we will go to obtain cocycles.

Remark 5.6. That this works we first have to verify that the $\widehat{\gamma}$ which we will choose is indeed a differential, and then that the γ_C fulfills the Lie algebra cocycle conditions. Also note that we do not claim that the parameterization is 1:1.

It is well-known that

$$\dim \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}) = \begin{cases} 2g, & \#A = 0,1, \\ 2g + (N-1), & \#A = N \geq 2\,. \end{cases} \tag{5.11}$$

Generators for this vector space are given by the $2g$ standard symplectic cycles and the cycles given by "circles" around the points $P_i \in A$. If $N \geq 1$ there is exactly one relation between these generators. By the splitting $A = I \cup O$ we fix a separating cycle class $[C_S]$. This is a non-vanishing element of $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$. It will be a preferable element to be taken as one of the basis elements as it will yield for our cocycles the fact that it will be local (see below).

For genus zero and $N \geq 1$ we have

$$\dim \mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C}) = N - 1. \tag{5.12}$$

A basis of the space is given by circles C_i around the points P_i where we leave out one of them. For example we can take $[C_i]$, $i = 1, \ldots, N-1$. We have the relation

$$\sum_{i=1}^{N-1} [C_i] = -[C_N]. \tag{5.13}$$

But there is a better choice. As explained above after choosing a splitting with separating cycle $[C_S]$ we take it as one of the basis elements and $N-2$ other $[C_i]s$ which are linearly independent.

For the standard splitting with $P_N = \{\infty\}$ we have

$$[C_S] = -[C_\infty], \qquad [C_i],\ i = 1, \ldots, N-2. \tag{5.14}$$

Integration around the C_i can be done via calculations of residues. Hence we always get for our geometric cocycles (for the standard splitting)

$$\gamma_{[C_S]} = \sum_{i=1}^{N-1} \mathrm{res}_{P_i}(\widehat{\gamma}) = -\,\mathrm{res}_\infty(\widehat{\gamma}), \qquad \gamma_{[C_i]} = \mathrm{res}_{P_i}(\widehat{\gamma}),\ i = 1, \ldots, N-2. \tag{5.15}$$

In the following we will define geometric cocycles for all our algebras. Results of the author [50], [51] shows that for those cocycles the $\gamma_{[C_S]}$ will be local, whereas the other $\gamma_{[C_i]}$ needed for a basis will not be local. Hence only the first type will allow us to extend the almost-grading. For simplicity we will sometimes use γ_i for $\gamma_{[C_i]}$ and γ_S for $\gamma_{[C_S]}$.

5.4. **Function algebra $\mathcal{A}$.** As $\mathcal{A}$ is an abelian Lie algebra all anti-symmetric bilinear forms will define a 2-cocycle. Moreover, there will be no non-trivial coboundaries. Hence $\mathrm{H}^2(\mathcal{A}, \mathbb{C}) \equiv \bigwedge^2 \mathcal{A}$, which is an infinite-dimensional vector space.

Definition 5.7. A cocycle γ for $\mathcal{A}$ is called
(a) *$\mathcal{L}$-invariant* if and only if

$$\gamma(e\,.\,f, g) + \gamma(f, e\,.\,g) = 0, \quad \forall f, g \in \mathcal{A}, \forall e \in \mathcal{L}, \tag{5.16}$$

(b) multiplicative if and only

$$\gamma(fg, h) + \gamma(gh, f) + \gamma(hf, g) = 0, \qquad f, g, h \in \mathcal{A}. \tag{5.17}$$

The definitions look rather ad-hoc for the moment, but we will find the explanation of $\mathcal{L}$-invariance when we discuss the differential operator algebra. Multiplicativity is needed for the current algebra. Note that the fact that γ is a multiplicative cocycle for the commutative algebra $\mathcal{A}$ can also be formulated as that it is a 1-cocycle in Connes's cyclic cohomology $\mathrm{HC}^1(\mathcal{A},\mathbb{C})$.

We define the bilinear map

$$\mathcal{A}\times\mathcal{A}\to\mathcal{F}^1,\qquad \widehat{\gamma}^{\mathcal{A}}(f,g)=f\cdot dg, \tag{5.18}$$

and the associated candidate for a cocycle

$$\gamma^{\mathcal{A}}_C(f,g)=\frac{1}{2\pi\mathrm{i}}\int_C f dg. \tag{5.19}$$

Proposition 5.8. *(1) $\gamma^{\mathcal{A}}_C$ is an $\mathcal{L}$-invariant cocycle.*
(2) $\gamma^{\mathcal{A}}_C$ is a multiplicative cocycle.
(3) The $\gamma^{\mathcal{A}}_{C_i}$ $i=1,\dots,N-1$ define linearly independent cocycles and hence linearly independent cohomology classes.

Proof. For the statements (1) and (2) see [50] and [55, Prop. 6.20]. With respect to the standard splitting the cocycles $\gamma^{\mathcal{A}}_{C_i}$ are exactly those $\mathcal{L}$-invariant which are bounded. In particular in the above references it is shown, that they are linearly independent. □

Proposition 5.9. [50], [55] *Given a splitting $A=I\cup O$ and the induced almost-grading, then*
(1) up to scaling

$$\gamma^{\mathcal{A}}_{C_S}(f,g)=\frac{1}{2\pi\mathrm{i}}\int_{C_S} f dg \tag{5.20}$$

is the unique $\mathcal{L}$-invariant cocycle which is local with respect to the grading.
(2) A basis of those $\mathcal{L}$-invariant cocycles which are bounded with respect to the almost-grading are given by the $\gamma^{\mathcal{A}}_{C_i}$, $P_i\in I$.

Proposition 5.10. [50], [55] *(1) The statements of Proposition 5.9 are true also for multiplicative cocycles.*
(2) Every bounded cocycle which is $\mathcal{L}$-invariant is multiplicative and vice versa.

The identifications of both types are done individually by reducing them to expressions as sums of $\gamma^{\mathcal{L}}_{C_i}$. Those have both properties.

The propositions above are true for all genera. Note that by calculating residues it is possible to calculate the values. As an illustration we will do this in Section 6.

Theorem 5.11. *Let γ be an $\mathcal{L}$-invariant or multiplicative cocycle for the multi-point function algebra in genus zero, then*
(1) γ is a linear combination of geometric cocycles of the type

$$\gamma^{\mathcal{A}}_i(f,g)=\frac{1}{2\pi\mathrm{i}}\int_{C_i} f dg=\mathrm{res}_{a_i}(f dg),\quad i=1,\dots,N-1. \tag{5.21}$$

(2) γ is bounded from above (by zero) with respect to the almost-grading given by the standard splitting.
(3) Every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa.

Before we start with the proof we quote

Proposition 5.12. *In the classical situation $g=0$, $N=2$ every $\mathcal{L}$-invariant cocycle γ is multiplicative and vice versa. It is given by*

$$\gamma(f,g)=\alpha\cdot\frac{1}{2\pi\mathrm{i}}\int_C f dg=\alpha\cdot\mathrm{res}_0(f dg),\quad \alpha\in\mathbb{C}, \tag{5.22}$$

and C a circle around 0. Moreover,

$$\gamma(A_n, A_m) = \alpha \cdot (-n) \cdot \delta_m^{-n}. \tag{5.23}$$

In particular γ is local and bounded by zero.

Proof. [55, Prop. 6.50, Rem. 6.64], [50]. □

Next we present some general techniques which will be used also in the proofs of the related statements for the other algebras. We consider the standard splitting. Recall that this says

$$\{P_1, P_2, \dots, P_{N-1}\} \cup \{P_N = \infty\}, \tag{5.24}$$

where P_i is the point given by the coordinate $a_i \in \mathbb{C}$. The function algebra $\mathcal{A}$ decomposes with respect to the basis elements

$$\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n, \qquad \mathcal{A}_n := \langle A_{n,1}, \dots, A_{n,N-1}\rangle_{\mathbb{C}}. \tag{5.25}$$

We introduce the associated filtration

$$\mathcal{A}_{(n)} = \bigoplus_{m\ge n} \mathcal{A}_m. \tag{5.26}$$

In [55], [49] we showed that

$$\mathcal{A}_{(n)} = \{f \in \mathcal{A} \mid \mathrm{ord}_{P_i}(f) \ge n,\ i = 1, \dots, N-1\}. \tag{5.27}$$

In particular, $\mathcal{A}_{(0)}$ is the subalgebra of meromorphic functions which are holomorphic on the affine part. We already introduced the elements $A_n^{(i)} := (z - a_i)^n, \quad n \in \mathbb{Z}$. Recall that beside the KN type basis elements the vector space $\mathcal{A}_{(0)}$ can also be generated by

$$\mathcal{B}_i = \{A_n^{(i)} \mid n \ge 0\} \tag{5.28}$$

for any fixed $i = 1, \dots, N-1$.

Proposition 5.13.

$$\begin{aligned} \mathcal{A}_{(n)} &= \bigcap_{i=1,\dots,N-1} \langle A_k^{(i)} \mid k \ge n\rangle_{\mathbb{C}}, \qquad \textit{for } n \ge 0 \\ \mathcal{A}_{(n)} &= \sum_{i=1}^{N-1} \langle A_k^{(i)} \mid k \ge n\rangle_{\mathbb{C}}, \qquad \textit{for } n < 0\ . \end{aligned} \tag{5.29}$$

Proof. Let $n \ge 0$. Then the elements in the intersection fulfill the order prescription $\mathrm{ord}_{P_i}(f) \ge n$. Vice versa every KN type basis elements lying in $\mathcal{A}_{(n)}$ can be expressed as linear combinations of powers $(z - a_i)^k$ with $k \ge n$ (just take the Taylor expansion at a_i). For negative n again the elements from the sum on the r.h.s. fulfill the order description, as $\mathrm{ord}_{P_j}(A_k^{(i)}) = 0 > n$ for $j \ne i$ and $\mathrm{ord}_{P_i}(A_k^{(i)}) \ge n$ for $k \ge n$. By the expansion into partial fractions the KN type basis elements on the l.h.s. are linear combinations of the elements on the r.h.s. Hence equality. □

We point out that the vector space sum above will be not a direct sum (at least if $N > 2$). The corresponding statements are of course true also for the spaces $\mathcal{F}^\lambda$, as for them the basis can be identified with the basis of $\mathcal{A}$ up to a λ-depending shift.

Proof. (of Theorem 5.11 Let γ be either $\mathcal{L}$-invariant or multiplicative. The statement of Proposition 5.12 will be valid for all $A_n^{(i)}$, $(i = 1, \ldots, N-1)$. We consider the values of $\gamma(\mathcal{A}_{(m)}, \mathcal{A}_{(m')})$ for $m + m' > 0$ and will show that they will vanish. Necessarily either m or m' has to be > 0. We assume that $m \geq 1 > 0$, in particular $f \in \mathcal{A}_{(m)}$ will be a linear combination of elements $A_k^{(i)}$ with $k \geq 1$ (see Proposition 5.13) for every i. If $m' \geq 0$ then $g \in \mathcal{A}_{(m')}$ will be a linear combination of $A_k^{(i)}$ and Proposition 5.12 shows that indeed

$$\gamma(\mathcal{A}_{(m)}, \mathcal{A}_{(m')}) = 0. \tag{5.30}$$

It remains to consider $m' < 0$. In this case let $f \in \mathcal{A}_{(m)}$ as above and take $A_k^{(i)}$, $k \geq m'$ for an arbitrary i. With respect to this i we write $f = \sum_{r \geq m} \alpha_r A_r^{(i)}$. Hence,

$$\gamma(f, A_k^{(i)}) = \gamma(\sum_{r \geq m} \alpha_r A_r^{(i)}, A_k^{(i)}) = \sum_{r \geq m} \alpha_r \cdot \gamma(A_r^{(i)}, A_k^{(i)}) = 0, \tag{5.31}$$

by Proposition 5.12 as $r + k \geq m + m' > 0$. This is true for all $i = 1, \ldots, N-1$, hence we get (5.30) too. In this way we showed that every $\mathcal{L}$-invariant or multiplicative cocycle is bounded with respect to the almost-grading given by the standard splitting. For bounded cocycles of this type the author showed in [50], see also [55], that they are geometric cocycles of the claimed form. In particular, both types of cocycles coincide. □

Corollary 5.14. *In the N-point genus zero situation the space of $\mathcal{L}$-invariant (or multiplicative) cocycles is $(N-1)$ dimensional and is isomorphic to $\mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C})$ via*

$$[C] \mapsto \gamma_C^{\mathcal{A}}; \qquad \gamma_C^{\mathcal{A}}(f, g) = \frac{1}{2\pi \mathrm{i}} \int_C f dg. \tag{5.32}$$

In particular, every $\mathcal{L}$ invariant cocycle or multiplicative cocycle is geometric.

The following is also part of the above classification.

Proposition 5.15. *With respect to the standard splitting up to rescaling the cocycle*

$$\gamma_\infty^{\mathcal{A}} = -\sum_{i=1}^{N-1} \gamma_{C_i}^{\mathcal{A}}(f, g) = \mathrm{res}_\infty(f dg) \tag{5.33}$$

is the unique $\mathcal{L}$-invariant (and equivalently multiplicative) cocycle which is local.

Remark 5.16. Let I be a non-empty subset of $\{1, 2, \ldots, N\}$ then the cocycle $\gamma_I^{\mathcal{A}} : \sum_{i \in I} \gamma_{C_i}^{\mathcal{A}}$ can be made to a local cocycle by taking the splitting I and $\{1, 2, \ldots, N\} \setminus I$ and choosing the induced almost-grading.

Remark 5.17. *(Heisenberg algebra.)* Of course, there does not exists a universal central extension of $\mathcal{A}$. But we could consider the algebra with central terms coming from the $\mathcal{L}$-invariant or equivalently multiplicative ones. The corresponding central extension of $\mathcal{A}$ will have a $(N-1)$-dimensional center and will be given as

$$[\widehat{f}, \widehat{g}] = \sum_{i=1}^{N-1} \alpha_i \cdot \gamma_{C_i}^{\mathcal{A}}(f, g) \cdot t_i, \quad \alpha_i \in \mathbb{C}, \quad [t_i, \widehat{\mathcal{A}}] = 0\,. \tag{5.34}$$

The local cocycle $\gamma_{C_S}^{\mathcal{A}}$ will yield a one-dimensional central extension which is almost-graded. One might either call the $(N-1)$-dimensional central extension or the almost-graded one-dimensional central extension *(multi-point) Heisenberg algebra.*

Conjecture 5.18. *For the higher genus and multi-point situation and γ a cocycle for $\mathcal{A}$ the following statements are equivalent:*

(1) *γ is $\mathcal{L}$-invariant*
(2) *γ is multiplicative*
(3) *γ is a geometric cocycle.*

Proposition 5.8 shows that Condition 3 implies the other two. But for the opposite the proofs presented above in genus zero will not work as there are geometric cocycles which are not bounded.

5.5. **Current and affine algebras.** First note that if $\mathfrak{g}$ is a perfect Lie algebra, i.e. $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, then $\overline{\mathfrak{g}}$ is perfect too. Hence, if $\mathfrak{g}$ is semi-simple the current algebra $\overline{\mathfrak{g}}$ admits a universal central extension. For current algebras introduced in Section 3.7 associated to a finite-dimensional Lie algebra $\mathfrak{g}$ geometric 2-cocycles can be given in the following way. First we fix β a symmetric, invariant, bilinear form for $\mathfrak{g}$. In particular $\beta([x,y],z)=\beta(x,[y,z])$. Then

$$\gamma^{\overline{\mathfrak{g}}}_{\beta,C}(x\otimes f,y\otimes g)=\beta(x,y)\cdot\gamma^{\mathcal{A}}_{C}(f,g)=\beta(x,y)\cdot\frac{1}{2\pi\mathrm{i}}\int_C fdg \tag{5.35}$$

defines a geometric cocycle for $\overline{\mathfrak{g}}$. Here the multiplicativity of $\int_C fdg$ is crucial.

If $\mathfrak{g}$ is simple then β is necessarily a multiple of the Cartan-Killing form. In this case Kassel [29], [28] proved that the algebra $\overline{\mathfrak{g}}=\mathfrak{g}\otimes\mathcal{A}$ for any commutative algebra $\mathcal{A}$ admits a universal central extensions which is given by

$$\widehat{\mathfrak{g}}^{univ}=\overline{\mathfrak{g}}\oplus(\Omega^1_{\mathcal{A}}/d\mathcal{A}) \tag{5.36}$$

with Lie structure

$$[x\otimes f,y\otimes g]=[x,y]\otimes fg+\beta(x,y)\overline{fdg}. \tag{5.37}$$

Here $\Omega^1_{\mathcal{A}}/d\mathcal{A}$ denotes the vector space of Kähler differentials of the algebra $\mathcal{A}$, and $\overline{fdg}$ means fdg modulo exact differentials. As in our situation $\Sigma_g\setminus A$ is an affine curve using Grothendieck's algebraic deRham theorem (see also Bremner [5] [7] and the author [51]) we can dualize the space by integrating $\overline{fdg}$ over closed curves C. In this way we identify the center with $\mathrm{H}_1(\Sigma_g\setminus A,\mathbb{C})$. More precisely, let $[C^{(k)}]$, $k=1,\ldots,2g+N-1$ a basis of the this homology space represented by cycles, then the universal central extension is given via geometric cocycles as

$$[x\otimes f,y\otimes g]=[x,y]\otimes fg+\beta(x,y)\cdot\sum_{i=1}^{2g+N-1}\alpha_i\cdot\frac{1}{2\pi\mathrm{i}}\int_{C^{(i)}}fdg\cdot t_i,\quad \alpha_i\in\mathbb{C} \tag{5.38}$$

with t_i central.

Now we return to the genus zero case where all geometric cocycles can be written with the help of residues. [3]

Proposition 5.19. *The universal central extension has a $(N-1)$-dimensional center and is given as*

$$[x\otimes f,y\otimes g]=[x,y]\otimes fg+\beta(x,y)\cdot\sum_{i=1}^{N-1}\alpha_i\cdot\mathrm{res}_{a_i}(fdg)\cdot t_i,\quad \alpha_i\in\mathbb{C} \tag{5.39}$$

with t_i central.

The one-dimensional central extensions are given (up to equivalence) as

$$[x\otimes f,y\otimes g]=[x,y]\otimes fg+\beta(x,y)\cdot\left(\sum_{i=1}^{N-1}\alpha_i\,\mathrm{res}_{a_i}(fdg)\right)\cdot t,\quad \alpha_i\in\mathbb{C}. \tag{5.40}$$

[3]Note that this is not the case for higher genus.

It is shown in [50], [55, Thm.9.2] that the cocycle class γ is local if and only if it can be represented by

$$\alpha \cdot \beta(x,y) \cdot \sum_{P \in I} \operatorname{res}_P(f dg). \tag{5.41}$$

By requiring $\mathcal{L}$-invariance for γ in the sense of [55, Def.9.11]:

$$\forall x, y \in \mathfrak{g}, \quad e \in \mathcal{L},\ g, h \in \mathcal{A}: \quad \gamma(x \otimes (e.g), y \otimes h) + \gamma(x \otimes g, y \otimes (e.h)) = 0 \tag{5.42}$$

we obtain even equality in (5.41) (not only up to equivalence).

Based on (5.39) it is easy to calculate central extensions as everything is defined in terms of the central extensions for $\mathcal{A}$ (which is done by calculating residues of rational function in the case of genus zero) and the values of the Cartan-Killing form.

In Appendix B we give the results in detail for the three-point situation and the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on the basis of our results in Section 6. See also Bremner [7] for explicit calculations in the 4-point situation using Gegenbauer polynomials.

Remark 5.20. The corresponding classification results are true for $\mathfrak{g}$ semisimple. In this case β has to be replaced by sums of the Cartan–Killing forms of the components. As $\overline{\mathfrak{g}}$ is also perfect it admits a universal central extension with a center of dimension $L \cdot (N-1)$ where L is the number of simple factors. See the above references for detailed results. In the reductive, but not semi-simple case, $\overline{\mathfrak{g}}$ does not have a universal central extension, nevertheless their exists classification results for cocycles which are $\mathcal{L}$-invariant if restricted to the abelian part. If k equals the dimension of the abelian part, then the dimension of the center is

$$\big(L + \frac{k(k+1)}{2}\big)(N-1). \tag{5.43}$$

5.6. **Vector field algebra.** For the vector field algebra $\mathcal{L}$ we choose the following bilinear map

$$\widehat{\gamma}^{\mathcal{L}}_R(e,f) = (\frac{1}{2}(ef''' - ef''') - R(ef' - e'f)dz. \tag{5.44}$$

Here again we identified the vector fields with their local representing functions and the element R is a projective connection (holomorphic outside of A), see Appendix C for its definition. Only by the term coming with R in it will be a well-defined differential. The associated geometric cocycles are given by

$$\gamma^{\mathcal{L}}_{C,R} = \frac{1}{2\pi \mathrm{i}} \int_C \widehat{\gamma}^{\mathcal{L}}, \qquad [C] \in \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}). \tag{5.45}$$

Proposition 5.21. [47],[55] *(1) The bilinear form $\gamma^{\mathcal{L}}_{C,R}$ is a 2-cocycle for $\mathcal{L}$.*
(2) A different choice of the projective connection will yield a cohomologous cocycle.

Let us quote the general result.

Theorem 5.22. *If $N \geq 1$ then $\mathcal{L}$ admits a universal central extension with a $(2g + N - 1)$ dimensional center. More precisely,*

$$\mathrm{H}^2(\mathcal{L},\mathbb{C}) \cong \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}) \tag{5.46}$$

where the isomorphism is given by $[C] \mapsto \gamma^{\mathcal{L}}_C$.

For the proof we refer to Skryabin [60]. It is related to the Novikov conjecture. For more details see also [55, §6.10.2].

Here we present in genus zero a direct and elementary proof of the above theorem which has the advantage of providing additional information. First with respect to local coordinates which are related via projective linear transformations we can choose $R \equiv 0$ on all such coordinate patches. This condition is fulfilled with respect to our standard covering of $\mathbb{P}^1(\mathbb{C})$. Hence, we might ignore the additional term.

Furthermore, the calculation efforts can be reduced by

Lemma 5.23. *For $a \in \Sigma_0$*

$$1/2\,\mathrm{res}_a((ef''' - e'''f)dz) = \mathrm{res}_a(ef'''dz) = -\,\mathrm{res}_a(e'''fdz)\,. \tag{5.47}$$

Proof. Using the rules for differentiation of products we see

$$(ef)''' = e'''f + 3e''f' + 3e'f'' + ef''' = e'''f + ef''' + 3(e'f')'. \tag{5.48}$$

As derivatives of functions do not have residues $\mathrm{res}_a(e'''f) = -\,\mathrm{res}_a(ef''')$. Hence the claim. □

Theorem 5.24. *Every cocycle γ for the algebra $\mathcal{L}$ in the genus zero, multi-point case is cohomologous to a bounded one with respect to the almost-grading given by the standard splitting.*

Proof. We recall the structure of the proof of Proposition 5.13. We have to take into account that the degree is shifted. In particular $\mathcal{L}_{(-1)}$ corresponds to those vector fields which are holomorphic on the affine part. Note that we have the generator set (which is not a basis)

$$e_n^{(i)} = (z - a_i)^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z},\ i = 1, \dots, N-1. \tag{5.49}$$

For each i separately, this is the usual Witt algebra and we have

$$[e_n^{(i)}, e_m^{(i)}] = (m-n)e_{n+m}^{(i)}, \qquad [e_0^{(i)}, e_m^{(i)}] = m \cdot e_m^{(i)}\,. \tag{5.50}$$

We start with an arbitrary cocycle γ and make cohomological changes. To this end we define a linear map $\psi : \mathcal{L} \to \mathbb{C}$ and modify γ by the corresponding coboundary $d_1\psi$. We prescribe ψ on a set of basis elements. For this we single out $i = 1$ and take as basis of $\mathcal{L}$

$$e_n^{(1)},\ n \geq -1, \qquad e_m^{(i)},\ m \leq -2,\ i = 1, \dots, N-1\,. \tag{5.51}$$

We set

$$\begin{aligned}
\psi(e_n^{(1)}) &:= \frac{1}{n}\gamma(e_0^{(1)}, e_n^{(1)}), \quad n = -1, 1, 2, ...\\
\psi(e_0^{(1)}) &:= \frac{1}{2}\gamma(e_{-1}^{(1)}, e_1^{(1)}),\\
\psi(e_n^{(i)}) &:= \frac{1}{n}\gamma(e_0^{(i)}, e_n^{(i)}), \quad n \leq -2\,.
\end{aligned} \tag{5.52}$$

The cohomologous cocycle is now

$$\gamma'(e,f) = \gamma(e,f) - \psi([e,f]). \tag{5.53}$$

By construction we obtain

$$\gamma'(e_0^{(1)}, e_n^{(1)}) = 0, \quad \forall n, \qquad \gamma'(e_0^{(i)}, e_n^{(i)}) = 0, \quad \forall n \leq -2,\ \forall i, \qquad \gamma'(e_{-1}^{(1)}, e_1^{(1)}) = 0\,. \tag{5.54}$$

To avoid cumbersome notation we use γ again for the cohomologous cocycle γ'. We write down the cocycle condition for each i:

$$\gamma([e_l^{(i)}, e_k^{(i)}], e_0^{(i)}]) + \gamma([e_k^{(i)}, e_0^{(i)}], e_l^{(i)}]) + \gamma([e_0^{(i)}, e_l^{(i)}], e_k^{(i)}]) = 0\,. \tag{5.55}$$

This evaluates to

$$(k-l)\gamma(e_{k+l}^{(i)}, e_0^{(i)}) - (k+l)\gamma(e_k^{(i)}, e_l^{(i)}) = 0. \tag{5.56}$$

First we consider pairs of elements which are holomorphic in the affine part, i.e. the space $\mathcal{L}_{(-1)} \times \mathcal{L}_{(-1)}$ and show

$$\gamma(f,g) = 0, \qquad f,g \in \mathcal{L}_{(-1)}. \tag{5.57}$$

By (5.54) $\gamma(e_n^{(1)}, e_0^{(1)}) = 0$ for all n and (5.56) implies $\gamma(e_k^{(1)}, e_l^{(1)}) = 0$ for $l \neq -k$. This shows (5.57) with the exception of $\gamma(e_1^{(1)}, e_{-1}^{(1)}) = 0$, which is true by our cohomological changes (5.54). In the next step we show

$$\gamma(e_k^{(i)}, e_l^{(i)}) = 0, \quad \text{for} \quad k+l > 0. \tag{5.58}$$

As long as $k,l \geq -1$ both vector fields are holomorphic and the claim is true by (5.57). Now assume $l \leq -2$. For $k+l > 0$ to be true $k \geq 3$ is necessary. In particular both $(k-l)$ and $(k+l) \neq 0$ and $e_{k+l}^{(i)}$ is holomorphic. From (5.56) the Equation (5.58) follows. [4] For the claim of the theorem we have to show that if $f \in \mathcal{L}_{(n)}$ and $g \in \mathcal{L}_{(m)}$ with $n+m > 0$ then $\gamma(f,g) = 0$. As $n+m > 0$ at least one of them has to be > 0. Assume $n > 0$. If $m \geq -1$ then both f and g are holomorphic hence the claim with (5.57). We assume $m \leq -2$. By Proposition 5.13 the elements $e_k^{(i)}$, $k \geq m$, $i = 1, \ldots, N-1$ generate $\mathcal{L}_{(m)}$. Hence g is a linear combination of those. Fix one i and one $e_k^{(i)}$ in this range. By Proposition 5.13 the corresponding $f \in \mathcal{L}_{(n)}$ can be written as a linear combination $\sum_l \alpha_l e_l^{(i)}$ of $e_l^{(i)}$ with $l \geq n \geq -m$. Now

$$\gamma(f, e_k^{(i)}) = \gamma(\sum_l \alpha_l e_l^{(i)}, e_k^{(i)}) = \sum_l \alpha_l \gamma(e_l^{(i)}, e_k^{(i)}) = 0, \tag{5.59}$$

by (5.58) as $l + k \geq n + m > 0$. As g is a linear combination of such $e_k^{(i)}$ the claim of the theorem follows. $\square$

Remark 5.25. In the proof above the prescription of (5.52) involving $n \leq -2$ was not necessary. We could have taken also the value 0 there. Nevertheless, the prescription given has the side-effect that the cohomological cocycle γ' restricted to the $N-1$ individual subalgebras $\langle e_n^{(i)} \mid n \in \mathbb{Z}\rangle$ which are copies of the Witt algebra, is there a multiple of the standard Virasoro cocycle (which is centered in degree zero). Here a word of warning is in order. We showed that the cocycle γ' is bounded from above by zero. It will not be true in general, that the γ' is also bounded from below by zero (meaning local). In general $\gamma'(e_n^{(i)}, e_m^{(j)}) \neq 0$ for $n,m \leq -2$. Moreover, by the results of [50] it will be bounded from below too if and only if there exists an $\alpha \in \mathbb{C}$ such that $\gamma' = \alpha \sum_{i=1}^{N-1} \gamma_i^{\mathcal{L}}$.

Theorem 5.26. *The cohomology space* $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$ *for the* $N-$*point genus zero vector field algebra* $\mathcal{L}$ *is* $(N-1)$*-dimensional. A basis is given by the cohomology classes*

$$[\gamma_i^{\mathcal{L}}] = [\gamma_{C_i}^{\mathcal{L}}], \quad i = 1, \ldots, N-1, \tag{5.60}$$

or equivalently

$$[\gamma_i^{\mathcal{L}}], \quad i = 1, \ldots, N-2, \qquad [\gamma_\infty^{\mathcal{L}}]\ . \tag{5.61}$$

In particular all cohomology classes are geometric cocycles. The cocycle $\gamma_\infty^{\mathcal{L}}$ *will be local with respect to the standard splitting.*

Proof. By Theorem 5.24 all cocycles are cohomologous to bounded cocycles with respect to the standard splitting. Hence the bounded cohomology classes constitute a basis of $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$. With respect to any splitting the bounded cocycle classes have been classified by the author in [50], see also [55]. In particular it is shown there that the noted cohomology classes are a basis of the

[4] Indeed, using the second relation in (5.54) we obtain (5.58) to be true for all $k + l \neq 0$, This statement is not needed in the proof, but see Remark 5.25.

bounded cocycles with respect to the standard splitting. In addition it is shown that the class $[\gamma_\infty^{\mathcal{L}}]$ is up to a scalar multiple the only cocycle which is local with respect to the standard splitting. $\square$

Theorem 5.27. *For $g = 0$ the algebra $\mathcal{L}$ is perfect, hence it admits a universal central extension generated by the above geometric cocycles.*

Proof. We know that $\mathcal{L}$ is generated by the $e_n^{(i)}$. As $[e_o^{(i)}, e_n^{(i)}] = n \cdot e_n^{(i)}$ and $[e_{-1}^{(i)}, e_1^{(i)}] = 2 \cdot e_0^{(i)}$ all generators can be written as commutators, i.e. $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. As a perfect Lie algebras $\mathcal{L}$ admits a universal central extensions parameterized by $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$. The universal central extension is freely generated by the cohomology classes given by the cocycles described above. $\square$

We recall that these cocycles can be explicitly calculated via residues, e.g.

$$
\begin{aligned}
\gamma_i^{\mathcal{L}}(e, f) &= \operatorname{res}_{a_i}(ef'''dz), \quad i = 1, 2, \ldots, N-2, \\
\gamma_\infty^{\mathcal{L}}(e, f) &= -\sum_{i=1}^{N-1} \operatorname{res}_{a_i}(ef'''dz) = \operatorname{res}_\infty(ef'''dz).
\end{aligned}
\tag{5.62}
$$

As above with respect to the standard splitting and almost-grading $\gamma_\infty^{\mathcal{L}}$ will be the unique local cocycle (up to equivalence and rescaling). The others $\gamma_i^{\mathcal{L}}$ are not local, but bounded from above. In Section 6 we will do some explicit calculations.

Remark 5.28. A description of the universal central extension was also given by Cox, Guo, Lu, and Zhao [12]. Their approach is again different. By intelligent guesses they prescribe for pairs of the basis elements from Section 4.7 $(N-1)$ bilinear maps and verify directly that these define 2-cocycles which are not coboundaries and are linearly independent and all cocycles can be written by them. For this quite involved combinatorial calculations are needed.

On the basis of [12] Jurisich and Martens treated the 3-point case [25] in detail by calculating the cocycles explicitly. Quite involved binomial identities are needed. We show in Section 6 how by consequently using the multi-point KN type algebra the cocycles can be calculated in a much simpler way.

5.7. **Differential operator algebra.** For the differential operator algebra $\mathcal{D}^1$ the cocycles of type (5.19) for $\mathcal{A}$ can be extended to $\mathcal{D}^1$ by zero on the subspace $\mathcal{L}$. For this we need that they are $\mathcal{L}$-invariant. The cocycles for $\mathcal{L}$ can be pulled back. In addition there is a third type of cocycles mixing $\mathcal{A}$ and $\mathcal{L}$:

$$
\gamma_{C,T}^{(m)}(e, g) := \frac{1}{2\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A},
\tag{5.63}
$$

with an affine connection T, with at most a pole of order one at a fixed point in O (see Appendix C for the definition of an affine connection)

Proposition 5.29. [50], [55] *$\gamma_{C,T}^{(m)}$ defines a 2-cocycle for $\mathcal{D}^1$. Another choice of a connection T will not change its cohomology class.*

In the above references we showed that every cocycle of $\mathcal{D}^1$ can be uniquely decomposed into a function algebra cocycle, a vector field cocycle and cocycle of mixing type. Also a complete description of bounded and local mixing cocycles is given. It follows exactly the same pattern as in the vector field case.

Remark 5.30. It is possible to choose $T = 0$ if all coordinate transformations are affine. But this is only true in genus 1. For different genera our T will be meromorphic. But it is possible to do with a pole of order one, which in our situation we put at the point ∞. In fact here we can do with $(T(z) = 0, T(w) = -2/w)$. Hence, in the affine part we do not see the appearance of T. For calculations in our genus zero with our system of coordinates we can completely ignore it (this is the same with a projective connection). After having verified that $\widehat{\gamma}^{(m)}(e,f) = (eg'' + Teg')dz$ is a well-defined differential it is uniquely given by the rational function representing it on the affine part. There the connection is equal to 0. The behaviour of the rational function at ∞ is given by the transformation law of 1-differentials, and we can calculate e.g. the residue there in the normal way.

For the genus zero situation again $[\gamma_i^{(m)}] = [\gamma_{C_i}^{(m)}]$ for $i = 1, \dots N-1$ will be a basis of the bounded mixing cocycles with respect to the standard splitting and $\gamma_\infty^{(m)} = \gamma_{C_S}^{(m)}$ the essentially unique local cocycle (up to rescaling and equivalence). In more detail

$$\begin{aligned} \gamma_i^{(m)}(e,g) &= \operatorname{res}_{a_i}(eg''dz), \quad i = 1, 2, \dots, N-2, \\ \gamma_\infty^{(m)}(e,g) &= -\sum_{i=1}^{N-1} \operatorname{res}_{a_i}(eg''dz) = \operatorname{res}_\infty(eg''dz). \end{aligned} \tag{5.64}$$

Theorem 5.31. *Every mixed cocycle for the differential operator algebra in the genus zero and multi-point situation is cohomologous to a bounded cocycle with respect to the standard splitting. Furthermore, this bounded cocycle is a linear combination of the geometric cocycles* $\gamma_i^{(m)}$, $i = 1, \dots, N-1$.

Proof. We will use the same strategy as above in the vector field algebra case. For the elements

$$e_n^{(i)} = (z-a_i)^{n+1}\frac{d}{dz}, \qquad A_n^{(i)} = (z-a_i)^n \tag{5.65}$$

we have the relations

$$[e_n^{(i)}, A_m^{(i)}] = e_n^{(i)} \,.\, A_m^{(i)} = m\, A_{n+m}^{(i)}. \tag{5.66}$$

In particular $[e_0^{(i)}, A_m^{(i)}] = m\, A_m^{(i)}$. We start with an arbitrary γ and modify it by a coboundary coming from a linear map $\psi : A \to \mathbb{C}$. We set

$$\begin{aligned} \psi(A_n^{(1)}) &= \frac{1}{n}\gamma(e_0^{(1)}, A_n^{(1)}), \quad n \geq 0, \\ \psi(A_0^{(1)}) &= \gamma(e_{-1}^{(1)}, A_1^{(1)}), \\ \psi(A_n^{(i)}) &= \frac{1}{n}\gamma(e_0^{(i)}, A_n^{(i)}), \quad n \leq -1, i = 1, \dots, N-1. \end{aligned} \tag{5.67}$$

By construction we have for the cohomologous cocycle $\gamma' = \gamma - d_1\psi$

$$\begin{gathered} \gamma'(e_0^{(1)}, A_n^{(1)}) = 0,\ \forall n \neq 0, \quad \gamma'(e_0^{(i)}, A_n^{(i)}) = 0,\ \forall n \leq -1,\ i = 2, \dots, N-1, \\ \gamma'(e_{-1}^{(1)}, A_1^{(1)}) = 0\ . \end{gathered} \tag{5.68}$$

We use again γ instead of γ'. Using the cocycle relation

$$\gamma([e_0^{(i)}, e_m^{(i)}], A_n^{(i)}) + \gamma([e_m^{(i)}, A_n^{(i)}], e_0^{(i)}) + \gamma([A_n^{(i)}, e_0^{(i)}], e_m^{(i)}) = 0 \tag{5.69}$$

we obtain

$$(m+n)\cdot\gamma(e_m^{(i)}, A_n^{(i)}) = n\cdot\gamma(e_0^{(i)}, A_{n+m}^{(i)}). \tag{5.70}$$

First we show that γ will vanish on holomorphic pairs, e.g. on $\mathcal{L}_{(-1)} \times A_{(0)}$. From (5.70) it follows that if $k + l \neq 0$ then $\gamma(e_k^{(i)}, A_l^{(i)}) = 0$. By the cohomological changes also $\gamma(e_{-1}^{(1)}, A_1^{(1)}) = 0$. From these properties we can conclude with nearly the same words as in the proof of Theorem 5.24 that

$$\gamma(f,g) = 0, \quad f \in \mathcal{L}_{(k)},\ g \in \mathcal{A}_{(l)},\ k + l > 0. \tag{5.71}$$

We do not reproduce these calculations here.

To conclude the proof of the theorem we use the classification results of the author on bounded cocycles from [50] with respect to the standard splitting and obtain the claimed representation. □

Proposition 5.32. *The differential operator algebra $\mathcal{D}^1$ in the genus zero and multi-point case is perfect.*

Proof. As in the vector field algebra case all generators are themselves commutators. This we showed above for the $e_n^{(i)}$. For the others we obtain

$$A_m^{(i)} = \frac{1}{m}[e_0^{(i)}, A_m^{(i)}],\ m \neq 0, \quad A_0^{(i)} = [e_{-1}^{(i)}, A_1^{(i)}]\ . \tag{5.72}$$

□

Theorem 5.33. *The differential operator algebra $\mathcal{D}^1$ in the genus zero and multi-point situation admits a universal central extension. It has a $3(N-1)$ dimensional center. The defining cocycles are given by the linearly independent cocycle classes of*

$$\gamma_i^{\mathcal{A}}, \quad \gamma_i^{\mathcal{L}}, \quad \gamma_i^{(m)}, \quad i = 1, \ldots, N-1, \tag{5.73}$$

or equivalently

$$\gamma_i^{\mathcal{A}}, \quad \gamma_i^{\mathcal{L}}, \quad \gamma_i^{(m)}, \quad i = 1, \ldots, N-2, \infty. \tag{5.74}$$

In the latter presentation the three cocycles $\gamma_\infty^{\mathcal{A}}, \gamma_\infty^{\mathcal{L}}, \gamma_\infty^{(m)}$ are the unique basis cocycles generating the 3-dimensional cohomology space yielding local classes with respect to the standard splitting.

Remark 5.34. Also for higher genus we have for every element from $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ three different type of cocycles for the differential operator algebra. With the help of them we can construct a central extension which has a $3 \cdot \dim \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ dimensional center. In accordance to the vector field case in the higher genus and the results in genus zero which we have just shown, it is reasonable to conjecture that this central extension is a universal central extension of $\mathcal{D}^1$.

5.8. **Central extensions of $\mathfrak{g}$-differential operator algebras.** Assume that $\mathfrak{g}$ is a simple Lie algebra. Then every cocycle γ of $\overline{\mathfrak{g}}$ can be made $\mathcal{L}$-invariant by cohomological changes. Such cocycle can be extended by zero on the complementary spaces involving elements form $\mathcal{L}$. Every cocycle of $\mathcal{L}$ defines a cocycle of $\mathcal{D}^1_{\mathfrak{g}}$ by pulling it back. In fact, every cocycle of $\mathcal{D}^1_{\mathfrak{g}}$ is cohomologous to a sum of these two types of cocycles. As both g and $\mathcal{L}$ are perfect, $\mathcal{D}^1_{\mathfrak{g}}$ is perfect too. Hence, it admits a universal central extension where the center is given by the two types of cocycles. From our results, presented here, respectively [51] it follows that this universal central extension has a center of dimension $2(N-1)$. The explicit cocycles were given above.

From the analysis in [51], [55] it follows that in the case that $\mathfrak{g}$ is semi-simple with L simple factors the universal central extension has a center of dimension $(L+1)(N-1)$. In the reductive but not semi-simple case the situation is a little bit more involved, as the general classification shows that mixing cocycles show up which come with a linear form on $\mathfrak{g}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$. Now we have to put $\mathcal{L}$-invariance in the conditions to obtain a complete classification. If k is the dimension of the abelian factor, then the center will have dimension

$$(L + \frac{k(k+1)}{2} + 1)(N-1). \tag{5.75}$$

Of course now it is not a universal extension anymore. See Remark B.2 for the example of $\mathfrak{gl}(n)$.

6. The three-point and genus zero case

6.1. **Symmetries.** The case of only three points where poles are allowed is to a certain extend special as we have additional symmetries. These symmetries can be used to simplify the calculations of structure constants even further.

Additionally, the three-point cases play a role in quite a number of applications. See e.g. the tetrahedron algebra appearing in statistical mechanics, in particular the work of Terwilliger and collaborators [23], [3].

By a complex automorphism of the Riemann sphere, i.e. by a fractional linear transformation, respectively by an PGL(2) action the three points can be brought to the points $0, 1$ and ∞. The corresponding automorphism will yield an isomorphism of the involved algebras. Even after this is done there are still automorphisms of $\mathbb{P}^1(\mathbb{C})$ permuting the 3 points $\{0, 1, \infty\}$. Hence, we still have the action of the symmetric group S_3 of 3 elements. The corresponding algebraic maps are now automorphisms of the algebras.

Recall that in the previous section we constructed for every splitting of the set A, here $\{0, 1, \infty\}$, into two disjoint non-empty subset I and O a distinguished basis which yields an almost-graded structure for the algebras. Essentially different splittings will yield essentially different basis elements respectively essentially different almost-gradings.

Here the only type of splitting is into a subset consisting of two points and a subset consisting of one point. After having fixed $A = \{0, 1, \infty\}$ we can by applying an automorphism from the remaining group S_3 such that

$$A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}. \tag{6.1}$$

This is exactly the situation which we will consider here.

Remark 6.1. I like to stress the fact, that this does not say, that there is only one possible choice of an almost-grading. In fact, given the set A and hence a unique algebra, we have 3 essentially different splitting, hence also 3 essentially different set of basis elements and consequently 3 almost-gradings. But in the three-point situation there will be always an automorphism of our algebra mapping the different almost-gradings to each other.

Remark 6.2. In [48] the situation $A = \{\alpha, -\alpha, \infty\}$ was considered for $\alpha \in \mathbb{C}$, $\alpha \neq 0$. This changes nothing. The corresponding algebras are isomorphic to the algebras considered here. Even our basis elements can be identified (up to some rescaling and re-indexing) and the structure equations remain the same (again up to scaling factors). The reason for the choice there was that we wanted to introduce a free parameter α which can be used to study degeneration process, respectively deformation families. For $\alpha \to 0$ we "degenerate" to the two-point situation. In this respect see also our joint work with Fialowski [16], [17], [18]. Clearly, everything that will be done in this section could be formulated also for $\{\alpha, -\alpha, \infty\}$, $\alpha \neq 0$.

6.2. **The associative algebra.** The basis elements of degree n of the algebra $\mathcal{A}$ with respect to our normalized splitting are (see Section 3, [46])

$$A_{n,1}(z) = z^n(z-1)^{n+1}, \qquad A_{n,2}(z) = z^{n+1}(z-1)^n, \quad n \in \mathbb{Z}, \tag{6.2}$$

where we ignore the scaling factors. We "symmetrize" and "anti-symmetrize" them for each degree separately by taking

$$\begin{aligned} A_n(z) = A_{n,2}(z) - A_{n,1}(z) \quad &= z^n(z-1)^n, \\ B_n(z) = A_{n,2}(z) + A_{n,1}(z) \quad &= z^n(z-1)^n(2z-1), \end{aligned} \tag{6.3}$$

Proposition 6.3. *The associative algebra $\mathcal{A}$ of meromorphic functions on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ holomorphic outside of* 0, 1, *and* ∞ *has as vector space basis*

$$\{A_n, B_n \mid n \in \mathbb{Z}\}, \tag{6.4}$$

with the structure equations

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned} \tag{6.5}$$

Proof. That the elements $A_{n,1}, A_{n,2}$ with $n \in \mathbb{Z}$ are a basis of $\mathcal{A}$ is a general fact by its very construction as Krichever–Novikov multi-point basis in [45] corresponding to our splitting of A. The transformation (6.3) is obviously a base change which even respects the homogeneous subspaces. By direct calculations which we moved to Appendix A we take from Lemma A.3 the structure equation. □

Our splitting introduces a (strong) almost-grading for the algebra $\mathcal{A}$

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n, \qquad \mathcal{A}_n = \langle A_n, B_n \rangle_{\mathbb{C}}, \quad \dim \mathcal{A}_n = 2. \tag{6.6}$$

This is clear from the general construction. But it can be easily illustrated by (6.5) as

$$\mathcal{A}_n \cdot \mathcal{A}_m \quad \subseteq \quad \mathcal{A}_{n+m} \oplus \mathcal{A}_{n+m+1}. \tag{6.7}$$

Next we want to study central extensions of $\mathcal{A}$ (considered as Lie algebras) which are given by geometric cocycles as introduced in Section 5. We showed that

$$\gamma_0^{\mathcal{A}}(f,g) = \operatorname{res}_0(fdg), \qquad \gamma_\infty^{\mathcal{A}}(f,g) = \operatorname{res}_\infty(fdg), \tag{6.8}$$

constitute a basis of the geometric cocycles. Recall that the set of geometric cocycles coincide with the $\mathcal{L}$-invariant respectively multiplicative cocycles. Note also that

$$\gamma_1^{\mathcal{A}} = \operatorname{res}_1(fdg) = -\gamma_0^{\mathcal{A}}(f,g) - \gamma_\infty^{\mathcal{A}}(f,g) \tag{6.9}$$

and that $\operatorname{res}_a(fdg) = -\operatorname{res}_a(gdf)$, $a \in \Sigma_0$. To calculate the cocycles it is enough to do the calculation for all type of pairs of basis elements A_n and B_m. The method is very simple. We use Lemma A.5 to express the integrand (e.g. $A_n A'_m dz$) in terms of A_k and B_k and then Lemma A.11, respectively Lemma A.8 and Lemma A.9 to calculate its value.

We start with

Proposition 6.4.

$$\begin{aligned} \gamma_\infty^{\mathcal{A}}(A_n, A_m) &= 2n\, \delta_m^{-n}, \\ \gamma_\infty^{\mathcal{A}}(A_n, B_m) &= 0, \\ \gamma_\infty^{\mathcal{A}}(B_n, B_m) &= 2n\delta_m^{-n} + 4(2n+1)\, \delta_m^{-n-1}\ . \end{aligned}$$

Proof. First we remark that in $A_n B'_m$ by Lemma A.5 only multiples of A_k show up and by Lemma A.11 their residues at ∞ will vanish. Hence the second relation. For the first relation we use $A_n A'_m dz = m B_{n+m-1} dz$. By Lemma A.11 there is only a non-vanishing value for the residue if $n + m - 1 = -1$ (equivalently $m = -n$) and it will be $-2m$. Hence the first relation. For $B_n B'_m dz$ we have 2 terms $2(2m+1)B_{m+n}$ and $m B_{n+m-1}$. The first term will have a residue if and only if $m = -n - 1$ and the second if and only if $m = -n$. This yields exactly the result of the third relation. □

Proposition 6.5.

$$\gamma_0^{\mathcal{A}}(A_n, A_m) = -n\,\delta_m^{-n},$$

$$\gamma_0^{\mathcal{A}}(A_n, B_m) = n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1}2^k\,\frac{(2k-3)!!}{k!}\delta_m^{-n-k}\,,$$

$$\gamma_0^{\mathcal{A}}(B_n, B_m) = -n\delta_m^{-n} - 2(2n+1)\,\delta_m^{-n-1}\,.$$

Proof. Relation 1 and 3 are automatic as the expressions fdg are the same as in the proof of Proposition 6.4 and the residues of B_k at 0 and ∞ are related via Lemma A.11. Hence only Relation 2 needs a calculation. The calculation can be made a little bit simpler by using the fact that $\operatorname{res}_a(A_n dB_m) = -\operatorname{res}_a(B_m dA_n)$. Using Lemma A.5 we get

$$\operatorname{res}_0(A_n B_m' dz) = -n(4\operatorname{res}_0(A_{n+m}dz) + \operatorname{res}_0(A_{n+m-1}dz)). \tag{6.10}$$

We set $k = -(n+m)$. From Lemma A.8 we conclude that there is no residue for $k < 0$. Next we consider $k \ge 2$ then

$$\begin{aligned} 4\operatorname{res}_0(A_{n+m}dz) + \operatorname{res}_0(A_{n+m-1}dz) &= (-1)^k\left(4\frac{(2k-3)!!}{(k-1)!}2^{k-1} - \frac{(2k-1)!!}{k!}2^k\right) \\ &= (-1)^k 2^k\,\frac{(2k-3)!!}{k!}. \end{aligned} \tag{6.11}$$

This gives the result in the generic situation. For the exceptional values for k we calculate the residues in (6.11) separately and use the values for the residue [5]

$$\operatorname{res}_0(A_0) = 0, \quad \operatorname{res}_0(A_{-1}) = -1, \quad \operatorname{res}_0(A_{-2}) = 2, \quad \operatorname{res}_0(A_{-3}) = -6, \tag{6.12}$$

to obtain for $k = 0$ the value -1 and for $k = 1$ the value -2. If we multiply these values with $-n$ we get the claimed values. □

Note that the second relation in the above proposition is expressed as a formal infinite sum. But for given n and m maximally one term will be non-zero. Hence, it has a well-defined value.

In accordance with the general results [50] about local cocycles only the cocycle $\gamma_\infty^{\mathcal{A}}$ is local. Here it will vanish for n, m outside of $-1 \le n+m \le 0$. Consequently, only the central extension defined via $\gamma_\infty^{\mathcal{A}}$ will admit an extension of the almost-grading to the central extension. Consequently, we obtain only in this case a triangular decomposition which is of importance for the representations appearing in field theory. This is not possible for the central extension defined by $\gamma_0^{\mathcal{A}}$.

Remark 6.6. If we change the almost-grading to the almost-grading introduced by the splitting $\{1,\infty\}\cup\{0\}$ the cocycle $\gamma_0^{\mathcal{A}}$ will become local. But we have to keep in mind that the distinguished basis will change automatically too.

6.3. **Current and affine algebra.** Recall that for a finite-dimensional Lie algebra $\mathfrak{g}$ the current algebra of KN-type is defined by $\overline{\mathfrak{g}} = \mathfrak{g}\otimes\mathcal{A}$. In particular every choice of a basis in $\mathfrak{g}$ and a basis in $\mathcal{A}$ will yield a basis of $\overline{\mathfrak{g}}$. For $\mathcal{A}$ we choose the basis A_n, B_n, $n \in \mathbb{Z}$ introduced above. Automatically we get an almost-graded structure of $\overline{\mathfrak{g}}$ induced by the splitting $\{0,1\}\cup\{\infty\}$. For its algebraic structure we obtain (via Proposition 6.3)

Proposition 6.7.

$$[x\otimes A_n, y\otimes A_m] = [x,y]\otimes A_{n+m},$$

$$[x\otimes A_n, y\otimes B_m = [x,y]\otimes B_{n+m},$$

$$[x\otimes B_n\cdot, y\otimes B_m = [x,y]\otimes(A_{n+m} + 4A_{n+m+1}).$$

[5] For further reference we give few more values.

In Section 3 we discussed its central extensions given by geometric cocycles

$$\gamma_\beta^{\overline{\mathfrak{g}}}(x\otimes f, y\otimes g) = \beta(x,y)\cdot\gamma^{\mathcal{A}}(f,g), \tag{6.13}$$

with $\beta(.,.)$ a symmetric, invariant bilinear form for $\mathfrak{g}$ and $\gamma^{\mathcal{A}}$ a multiplicative 2-cocycle for the algebra $\mathcal{A}$. Recall that if $\mathfrak{g}$ is a simple Lie there exists a universal central extension $\widehat{\mathfrak{g}}$. In our case it has a two-dimensional center and will be given by

$$\begin{aligned}[x\otimes f, y\otimes g] = [x,y]\otimes(f\cdot g) &+ \alpha_0\cdot\beta(x,y)\cdot\gamma_0^{\mathcal{A}}(f,g)\cdot t_0\\ &+ \alpha_\infty\cdot\beta(x,y)\cdot\gamma_\infty^{\mathcal{A}}(f,g)\cdot t_\infty\end{aligned} \tag{6.14}$$

with $\alpha_0,\alpha_\infty\in\mathbb{C}$, t_0,t_∞ central in $\widehat{\mathfrak{g}}$, and β the Cartan-Killing form. The values of the cocycles for the introduced basis elements have been calculated above and will not be repeated here. In accordance with the general results [51] the centrally extended current algebra will be an almost-graded extension of the current algebra with respect to this basis if and only if $\alpha_0 = 0$. It is an easy task to write everything explicitly for special cases of the Lie algebra $\mathfrak{g}$. We will give the result for $\mathfrak{sl}(2,\mathbb{C})$ in Appendix B.

As explained in Section 5 this classification results can be extended to arbitrary semi-simple Lie algebras (finite-dimensional). If we require "$\mathcal{L}$-invariance" for the cocycle even the reductive case is possible.

6.4. **Vector field algebra.** Recall that in the genus $g = 0$ case and $P_N = \infty$ the elements g for $\mathcal{F}^\lambda$ for $\lambda\in\frac{1}{2}\mathbb{Z}$ are given by

$$g(z) = a(z)dz^\lambda, \quad \text{with} \quad a(z)\in\mathcal{A}. \tag{6.15}$$

In particular, a basis of $\mathcal{A}$ induces a basis of $\mathcal{F}^\lambda$. As explained in Section 3 a λ-depending shift is convenient. We take as basis elements the elements

$$g_n^\lambda := A_{n-\lambda}dz^\lambda, \qquad h_n^\lambda := B_{n-\lambda}dz^\lambda, \quad n\in\mathbb{J}_\lambda. \tag{6.16}$$

The corresponding almost-grading reads as

$$\mathcal{F}^\lambda = \bigoplus_{n\in\mathbb{J}_\lambda}\mathcal{F}_n^\lambda, \qquad \mathcal{F}_n^\lambda = \langle g_n^\lambda, h_n^\lambda\rangle_{\mathbb{C}}. \tag{6.17}$$

For the vector field (i.e. forms of weight -1) we use

$$e_n := A_{n+1}\frac{d}{dz}, \qquad f_n := B_{n+1}\frac{d}{dz}, \quad n\in\mathbb{Z}. \tag{6.18}$$

The vector spaces $\mathcal{F}^\lambda$ are modules over $\mathcal{L}$. If $e = a\frac{d}{dz}$ and $g = bdz^\lambda$ then the module structure reads as

$$e\,.\,g = (a\cdot b' + \lambda\, b\cdot a')dz^\lambda. \tag{6.19}$$

With respect to our basis elements the structure equations are given by

Proposition 6.8.

$$\begin{aligned}
e_n\,.\,g_m^\lambda &= (m+\lambda n)\,h_{m+n}^\lambda,\\
e_n\,.\,h_m^\lambda &= (m+\lambda n)\,g_{m+n}^\lambda + (4(m+\lambda n)+2)\,g_{n+m+1}^\lambda,\\
f_n\,.\,g_m^\lambda &= (m+\lambda n)\,g_{m+n}^\lambda + (4(m+\lambda n)+2\lambda)\,g_{n+m+1}^\lambda,\\
f_n\,.\,h_m^\lambda &= (m+\lambda n)\,h_{m+n}^\lambda + (4(m+\lambda n)+2+2\lambda)\,h_{n+m+1}^\lambda.
\end{aligned}$$

Proof. We calculate the expression (6.19) for the pairs of basis elements. Then we use the expressions for the derivatives from Lemma A.4 and for the products from Lemma A.3 and obtain in a straight forward manner the results. □

For $\lambda=-1$ we get the vector field algebra structure.

Proposition 6.9.
$$
\begin{aligned}
[e_n,e_m]&=(m-n)\,f_{m+n},\\
[e_n,f_m]&=(m-n)\,e_{m+n}+(4(m-n)+2)\,e_{n+m+1},\\
[f_n,f_m]&=(m-n)\,f_{m+n}+4(m-n)\,f_{n+m+1}.
\end{aligned}
$$

These expressions clearly exhibit the almost-graded structure. Observe that the algebra of global holomorphic vector fields is the subalgebra

$$
\langle e_{-1},f_{-1},e_0\rangle_{\mathbb{C}}, \tag{6.20}
$$

which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Next we calculate the universal central extension. We know that it has a two-dimensional center and can be given as

$$
[\widehat{e},\widehat{f}]=\widehat{[e,f]}+\alpha_0\cdot\gamma_0^{\mathcal{L}}(e,f)\cdot t_o+\alpha_\infty\cdot\gamma_\infty^{\mathcal{L}}(e,f)\cdot t_\infty \tag{6.21}
$$

with

$$
\begin{aligned}
\gamma_0^{\mathcal{L}}(e,f)&=1/2\operatorname{res}_0(e\cdot f'''-f\cdot e''')dz=\operatorname{res}_0(e\cdot f''')dz=-\operatorname{res}_0(f\cdot e''')dz\\
\gamma_\infty^{\mathcal{L}}(e,f)&=1/2\operatorname{res}_\infty(e\cdot f'''-f\cdot e''')=\operatorname{res}_\infty(e\cdot f''')dz=-\operatorname{res}_\infty(f\cdot e''')dz\,.
\end{aligned} \tag{6.22}
$$

Here we used Lemma 5.23

First we consider the point ∞ and obtain in this way the local cocycle.

Proposition 6.10.
$$
\begin{aligned}
&\gamma_\infty^{\mathcal{L}}(e_n,e_m)=2(n^3-n)\,\delta_m^{-n}+4n(n+1)(2n+1)\delta_m^{-n-1}\\
&\gamma_\infty^{\mathcal{L}}(e_n,f_m)=0,\\
&\gamma_\infty^{\mathcal{L}}(f_n,f_m)=2(n^3-n)\,\delta_m^{-n}+8n(n+1)(2n+1)\delta_m^{-n-1}+8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}
\end{aligned}
$$

Proof. The method is the same as in the proof of Proposition 6.4. We have the expressions of e_n, respectively f_n via the basis functions A_n and B_n, use Lemma A.7 and build the integrand in one of the forms given in (6.22). Finally Lemma A.11 will give the residue. In the expression of A_nB_m''' and B_mA_n''' only linear combinations of A_k show up. But they do not have a residue at ∞. Hence, the second relation.

The others we have to calculate. First using Lemma A.7

$$
\begin{aligned}
\gamma_\infty^{\mathcal{L}}(e_n,e_m)&=-\operatorname{res}_\infty(A_{m+1}A_{n+1}''')\\
&=-(n+1)n(n-1)\operatorname{res}_\infty B_{n+m-1}-2(n+1)n(2n+1)\operatorname{res}_\infty B_{n+m}.
\end{aligned} \tag{6.23}
$$

Taking the residues at ∞ via Lemma A.11 we get exactly the first relation. For $\gamma_\infty^{\mathcal{L}}(f_n,f_m)=-\operatorname{res}_\infty(B_{m+1}B_{n+1}''')$ we obtain in completely in the same manner the last relation (now with three terms). □

Proposition 6.11.
$$
\begin{aligned}
\gamma_0^{\mathcal{L}}(e_n,e_m)&=-(n^3-n)\,\delta_n^{-m}-2n(n+1)(2n+1)\delta_m^{-n-1}\\
\gamma_0^{\mathcal{L}}(e_n,f_m)&=(n^3-n)\,\delta_m^{-n}+6n^2(n+1)\delta_m^{-n-1}+6n(n+1)^2\delta_m^{-n-2}\\
&\qquad+\sum_{k\ge 3}n(n+1)(n+k-1)(-1)^k2^k\cdot 3\cdot\frac{(2k-5)!!}{k!}\delta_m^{-n-k}\\
\gamma_0^{\mathcal{L}}(f_n,f_m)&=-(n^3-n)\,\delta_m^{-n}-4n(n+1)(2n+1)\delta_m^{-n-1}-4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,.
\end{aligned}
$$

Proof. In all cases the integrand is the same as the one considered in Proposition 6.10. Hence by, Lemma A.11 we obtain $-1/2$ of the values there for the 1. and 3. relation. It remains to calculate the 2nd relation. We have

$$\gamma_0^{\mathcal{L}}(e_n, f_m) = -\operatorname{res}_0(B_{m+1}A'''_{n+1}). \tag{6.24}$$

From Lemma A.7 we obtain for it

$$-(n+1)n\left(8(2n+1)\operatorname{res}_0 A_{n+m+1} + 2(4n-1)\operatorname{res}_0 A_{n+m} + (n-1)\operatorname{res}_0 A_{n+m-1}\right) \tag{6.25}$$

Next we have to consult Lemma A.8 for calculating the residues. We set $k = -(n+m)$. If $k > 0$ there will be no residue. For $k \geq 3$ we are in the generic situation and obtain for the last factor in (6.25)

$$\begin{aligned}(-1)^{-k-1}\frac{(2k-5)!!}{k!}2^k\Big(2(2n+1)k(k-1) - (4n-1)(2k-3)k + (n-1)(2k-1)(2k-3)\Big)\\ = (-1)^{-k-1}\frac{(2k-5)!!}{k!}2^k\cdot 3\cdot(n+k-1).\end{aligned} \tag{6.26}$$

This yields exactly the generic expression. For $k = 0, 1$ and 2 we have to take the residues in (6.25) individually and use the the values (6.12) and get exactly the claimed expressions. $\square$

See [25] for similar results.

Note also that we can express the term

$$\frac{(2k-5)!!}{k!}2^k\cdot 3 = \frac{12}{k(k-1)}\binom{2(k-2)}{k-2} \tag{6.27}$$

if this is more convenient.

Again only the cocycle $\gamma_\infty^{\mathcal{L}}$ will be local with respect to the almost-grading introduced by our splitting, respectively by our basis. Hence, only for its corresponding central extension we have a triangular decomposition. Everything what was said for the function algebra case, remains true here.

6.5. **Differential operator algebra $\mathcal{D}^1$.** Recall that $\mathcal{D}^1$ is the (Lie algebra) semi-direct sum of $\mathcal{A}$ with $\mathcal{L}$ where $\mathcal{L}$ operates on $\mathcal{A}$ by taking the derivative. The homogeneous subspace of degree n is now

$$\langle A_n, B_n, e_n, f_n\rangle_{\mathbb{C}}. \tag{6.28}$$

The subalgebra $\mathcal{A}$ is abelian and Proposition 6.9 gives the structure equations for the vector fields. Proposition 6.8 specialized for $\lambda = 0$ yields the the equations for the mixed terms.

Proposition 6.12.

$$\begin{aligned}
[e_n, A_m] &= -[A_m, e_n] = m\,B_{m+n},\\
[e_n, B_m] &= -[B_m, e_n] = m\,A_{m+n} + (4m+2)\,A_{n+m+1},\\
[f_n, A_m] &= -[A_m, f_n] = m\,A_{m+n} + 4m\,A_{n+m+1},\\
[f_n, B_m] &= -[B_m, f_n] = m\,B_{m+n} + (4m+2)\,B_{m+n+1},
\end{aligned}$$

The geometric cocycles yield a 6-dimensional central extension. In addition to the 4 basis elements given by the pure cocycles given by the Propositions 6.4, 6.5, 6.10, 6.11 we have two additional basis cocycles ($e \in \mathcal{L}, g \in \mathcal{A}$)

$$\gamma_0^{(m)}(e, g) = \operatorname{res}_0(eg''dz), \qquad \gamma_\infty^{(m)}(e, g) = \operatorname{res}_\infty(eg''dz). \tag{6.29}$$

Evaluated for the basis elements we obtain

Proposition 6.13.

$$
\begin{aligned}
&\gamma_\infty^{(m)}(e_n, A_m) = 0,\\
&\gamma_\infty^{(m)}(e_n, B_m) = -2n(n+1)\delta_m^{-n} - 4(n+1)(2n+1)\delta_m^{-n-1},\\
&\gamma_\infty^{(m)}(f_n, A_m) = -2n(n+1)\delta_m^{-n} - 4(n+1)(2n+3)\delta_m^{-n-1},\\
&\gamma_\infty^{(m)}(f_n, B_m) = 0\,.
\end{aligned}
$$

Proposition 6.14.

$$
\begin{aligned}
\gamma_0^{(m)}(e_n, A_m) &= -n(n+1)\delta_m^{-n} - 2(n+1)^2\delta_m^{-n-1}\\
&\qquad + \sum_{k\ge 2}(n+1)(n+k)(-1)^k 2^k \cdot \frac{(2k-3)!!}{k!}\delta_m^{-n-k}\,.\\
\gamma_0^{(m)}(e_n, B_m) &= n(n+1)\delta_m^{-n} + 2(n+1)(2n+1)\delta_m^{-n-1},\\
\gamma_0^{(m)}(f_n, A_m) &= n(n+1)\delta_m^{-n} + 2(n+1)(2n+3)\delta_m^{-n-1},\\
\gamma_0^{(m)}(f_n, B_m) &= -n(n+1)\delta_m^{-n} - 6(n+1)^2\delta_m^{-n-1} - 6(n+1)(n+2)\delta_m^{-n-2}\\
&\qquad + \sum_{k\ge 3}(n+1)(n+k)(-1)^{k-1} 2^k \cdot 3 \cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k}\,.
\end{aligned}
$$

Proof. (Common) The proofs of Proposition 6.13 and Proposition 6.14 can be done in a completely analogous way as for the vector field algebra. But things can be simplified by using the results for the function and vector field algebra and the simple fact $A_n' = nB_{n-1}$. Hence for C any of the elements we get

$$
C \cdot A_n'' = n\, C \cdot B_{n-1}', \quad C \cdot B_n'' = \frac{1}{n+1} C \cdot A_{n+1}''' \,, n \ne -1. \tag{6.30}
$$

Consequently, $(a = 0, \infty)$

$$
\begin{aligned}
\gamma_a^{(m)}(e_n, A_m) &= m \cdot \gamma_a^{\mathcal{A}}(A_{n+1}, B_{m-1}), & \gamma_a^{(m)}(f_n, A_m) &= m \cdot \gamma_a^{\mathcal{A}}(B_{n+1}, B_{m-1}),\\
\gamma_a^{(m)}(e_n, B_m) &= (1/(m+1)) \cdot \gamma_a^{\mathcal{L}}(e_n, e_m), & \gamma_a^{(m)}(f_n, B_m) &= (1/(m+1)) \cdot \gamma_a^{\mathcal{L}}(f_n, e_m).
\end{aligned} \tag{6.31}
$$

For the second line we exclude $m = -1$. By simple substitution we obtain exactly the claimed expressions. In fact even for $m = -1$ the results obtained by substitution are correct, but of course need to be verified directly. □

As shown in Section 5 the differential operator algebra admits a universal central extension and the introduced six geometric cocycles, each associated to a different central element will yield the universal central extension.

Proposition 6.15. *A cocycle class $[\gamma]$ for $\mathcal{D}^1$ will be local (with respect to the standard splitting) if and only if γ is cohomologous to a linear combination*

$$
\gamma \quad \sim \quad \alpha_1 \cdot \gamma_\infty^{\mathcal{A}} + \alpha_2 \cdot \gamma_\infty^{\mathcal{L}} + \alpha_3 \cdot \gamma_\infty^{(m)}, \qquad \alpha_i \in \mathbb{C}. \tag{6.32}
$$

In particular, the space of local cohomology classes is 3-dimensional.

This is a general result of [50], [55] which is for the 3-point illustrated by the above calculations. In the very general case (meaning arbitrary genus, arbitrary number of marked points, arbitrary splitting) the three cocycles in (6.32) are obtained by integrating over a separating cycle. By allowing suitable meromorphic affine and projective connections holomorphic outside of A we can even obtain equality in (6.32).

7. Lie superalgebras

Lie superalgebras of Krichever–Novikov type were discussed in [54], [55], see also Kreusch [30] and Leidwanger and Morier-Genoud [34]. Here we present the N-point genus zero situation. In particular we will also deal with the $N = 3$ case.

A Lie superalgebra $\mathcal{S}$ is a vector space which is decomposed into subspaces of even and odd elements $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, i.e. $\mathcal{S}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bilinear map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that for elements x, y of pure parity

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]. \tag{7.1}$$

Here $\bar{x}$ is the parity of x, etc. These conditions say that

$$[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subseteq \mathcal{S}_{\bar{0}}, \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{1}}, \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{0}}, \tag{7.2}$$

and that $[x, y]$ is symmetric for x and y odd, otherwise anti-symmetric. Now $\mathcal{S}$ is a *Lie superalgebra* if in addition the *super-Jacobi identity* (for x, y, z of pure parity)

$$(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0 \tag{7.3}$$

is valid.

The Lie superalgebra, we discuss here, is based on half-forms. In fact, it will be a superalgebra of Neveu-Schwarz type. Recall the associative product (6.5)

$$\cdot\ \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L}\ . \tag{7.4}$$

We introduce the vector space $\mathcal{S}$ with the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e, \varphi), (f, \psi)] := ([e, f] + \varphi \cdot \psi, e\,.\,\varphi - f\,.\,\psi). \tag{7.5}$$

The elements of $\mathcal{L}$ are denoted by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

The definition (7.5) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a super-bracket (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e, \varphi] := -[\varphi, e] := e\,.\,\varphi = (e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2} \tag{7.6}$$

and

$$[\varphi, \psi] := \varphi \cdot \psi\ . \tag{7.7}$$

The elements of $\mathcal{L}$ are the elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ the elements of odd parity.

Proposition 7.1. [55, Prop. 2.15] *The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Clearly, the vector field algebra $\mathcal{L}$ is a Lie subalgebra.

Remark 7.2. Recall that for genus $g \geq 1$ the choice of a square root of the canonical bundle, also called a theta characteristics, will be needed for fixing the super algebra. Such a choice corresponds to choosing a *spin structure* on Σ_g.

Central extensions of $\mathcal{S}$ are also of relevance and will be given in the following. The Lie algebra 2-cocycle conditions for γ have to be replaced by its super version, saying

$$\gamma(x, y) = -(-1)^{\bar{x}\bar{y}}\gamma(x, y). \tag{7.8}$$

and

$$(-1)^{\bar{x}\bar{z}}\gamma(x, [y, z]) + (-1)^{\bar{y}\bar{x}}\gamma(y, [z, x]) + (-1)^{\bar{z}\bar{y}}\gamma(z, [x, y]) \ = \ 0. \tag{7.9}$$

As usual our elements x, y, z should be of pure parity.

Proposition 7.3. [54] *Let C be any closed (differentiable) curve on Σ_g not meeting the points in A, and let R be any (holomorphic) projective connection then the bilinear extension of*

$$\begin{aligned}
\gamma^{\mathcal{S}}_{C,R}(e,f) &:= \frac{1}{2\pi\mathrm{i}}\int_C \left(\frac{1}{2}(ef'''-e'''f) - R\cdot(ef'-e'f)\right)dz \\
\gamma^{\mathcal{S}}_{C,R}(\varphi,\psi) &:= \frac{1}{2\pi\mathrm{i}}\int_C (\varphi''\cdot\psi + \varphi\cdot\psi'' - R\cdot\varphi\cdot\psi)\,dz \qquad (7.10)\\
\gamma^{\mathcal{S}}_{C,R}(e,\varphi) &:= 0
\end{aligned}$$

gives a geometric Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. A different projective connection will yield a cohomologous cocycle.

A similar formula was given by Bryant in [8]. By adding the projective connection in the second part of (7.10) he corrected some formula appearing in [4]. He only considered the two-point case and only the integration over a separating cycle. See also [30] for the multi-point case, where still only the integration over a separating cycle is considered.

In contrast to the differential operator algebra case the two parts cannot be prescribed independently. Only with the same integration path (more precisely, homology class) and the given factors in front of the integral it will work.

Now we consider the genus zero case. As shown above, all cohomology classes for the vector field algebra $\mathcal{L}$ are bounded classes and are obtained via integrating over C_i, hence by calculating residues. From the general theorem [54], [55] it follows that the cohomology classes for the superalgebra will be uniquely be given by taking the same point for calculating the residue.

For explicit calculations of the structure constants see Proposition 6.8 and Proposition 6.3. We will only give the results for the 3-point case using the notation of Section 6. In addition to the basis elements e_n and f_n of $\mathcal{L}$ we take

$$\varphi_n = A_{n+1/2}(dz)^{-1/2}, \quad \psi_n = B_{n+1/2}(dz)^{-1/2}, \qquad n\in\mathbb{Z}+1/2. \qquad (7.11)$$

Additionally to the structure constants of Proposition 6.9 we have

Proposition 7.4.

$$\begin{aligned}
[\varphi_n,\varphi_m] &= e_{n+m}\\
[\varphi_n,\psi_m] &= f_{n+m}\\
[\psi_n,\psi_m] &= e_{n+m} + 4\,e_{n+m+1}\\
[e_n,\varphi_m] &= (m-n/2)\,\psi_{n+m}\\
[e_n,\psi_m] &= (m-n/2)\,\varphi_{n+m} + (4m-2n+2)\,\varphi_{n+m+1}\\
[f_n,\varphi_m] &= (m-n/2)\,\varphi_{n+m} + (4m-2n-1)\,\varphi_{n+m+1}\\
[f_n,\psi_m] &= (m-n/2)\,\psi_{n+m} + (4m-2n+1)\,\psi_{n+m+1}\,.
\end{aligned}$$

Similar expressions are given by Leidwanger and Morier-Genoud [34, Prop. 3.8] with respect to a slightly different scaled system of basis elements also used in [48], [16].

Next we consider 2-cocycles. Again with respect to the standard system of coordinates $(z, w = 1/z)$ the connection $R=0$ can be chosen. We have a two-dimensional space of geometric cocycles generated by $\gamma_0^{\mathcal{S}}$ and $\gamma_\infty^{\mathcal{S}}$ obtained by taking the residue from (7.10) at 0 and ∞. For pairs of pure vector field type arguments we have the result of Proposition 6.10 and Proposition 6.11. For mixing of pure types it is zero. It remains to consider pairs of $-1/2$-forms. As

$$\operatorname{res}_a(\varphi''\psi+\varphi\psi'') = \operatorname{res}_a(\varphi''\psi+\varphi\psi''-(\varphi\psi')'+(\varphi'\psi)') = 2\operatorname{res}_a(\varphi''\psi) = 2\operatorname{res}_a(\psi''\varphi)$$

we have

$$\begin{aligned}\gamma_\infty^{\mathcal{S}}(\varphi,\psi) &= \mathrm{res}_\infty((\varphi''\psi+\varphi\psi'')\,dz) = 2\,\mathrm{res}_\infty(\varphi''\psi\,dz)\\ \gamma_0^{\mathcal{S}}(\varphi,\psi) &= \mathrm{res}_0((\varphi''\psi+\varphi\psi'')\,dz) = 2\,\mathrm{res}_0(\varphi''\psi\,dz).\end{aligned} \tag{7.12}$$

Note that these values are already calculated in the context for the mixed cocycle for the differential operator algebra. In fact we have with $\hat{n}=n+1/2$ and $\hat{m}=m+1/2$

$$\begin{aligned}\gamma_a^{\mathcal{S}}(\varphi_n,\varphi_m) &= 2\gamma_a^{(m)}(e_{\hat{n}-1},A_{\hat{m}}), \qquad &\gamma_a^{\mathcal{S}}(\varphi_n,\psi_m) &= 2\gamma_a^{(m)}(e_{\hat{n}-1},B_{\hat{m}}),\\ \gamma_a^{\mathcal{S}}(\psi_n,\varphi_m) &= 2\gamma_a^{(m)}(f_{\hat{n}-1},A_{\hat{m}}), \qquad &\gamma_a^{\mathcal{S}}(\psi_n,\psi_m) &= 2\gamma_a^{(m)}(f_{\hat{n}-1},B_{\hat{m}})\ .\end{aligned} \tag{7.13}$$

With these identities we recall from the results there

Proposition 7.5.

$$\begin{aligned}\gamma_\infty^{\mathcal{S}}(\varphi_n,\varphi_m) &= 0\\ \gamma_\infty^{\mathcal{S}}(\varphi_n,\psi_m) &= -4(n-1/2)(n+1/2)\delta_m^{-n} - 8n(2n+1)\delta_m^{-n-1}.\\ \gamma_\infty^{\mathcal{S}}(\psi_n,\psi_m) &= 0\end{aligned}$$

Proposition 7.6.

$$\begin{aligned}\gamma_0^{\mathcal{S}}(\varphi_n,\varphi_m) &= -2(n+1/2)(n-1/2)\delta_m^{-n} + 4(n+1/2)^2\delta_m^{-n-1}\\ &\qquad + 2\sum_{k\ge 2}(n+1/2)(n-1/2+k)(-1)^k 2^k\frac{(2k-3)!!}{k!}\delta_m^{-n-k}\end{aligned}$$

$$\gamma_0^{\mathcal{S}}(\varphi_n,\psi_m) = 2(n-1/2)(n+1/2)\delta_m^{-n} + 4n(2n+1)\delta_m^{-n-1}.$$

$$\begin{aligned}\gamma_0^{\mathcal{S}}(\psi_n,\psi_m) &= -2(n+1/2)(n-1/2)\delta_m^{-n} - 12(n+1/2)^2\delta_m^{-n-1} - 12(n+1/2)(n+3/2)\delta_m^{-n-2}\\ &\qquad + 2\sum_{k\ge 3}(n+1/2)(n-1/2+k)(-1)^{k-1}2^k\cdot 3\cdot\frac{(2k-5)!!}{k!}\delta_m^{-n-k}\ .\end{aligned}$$

We illustrated again that only the cocycle $\gamma_\infty^{\mathcal{S}}$ is local with respect to the standard splitting. The $\gamma_\infty^{\mathcal{S}}$ is the one which was considered by Kreusch [30] (up to a different indexing of the basis elements).

Remark 7.7. Here we considered the central element to be even. We could have even dropped this assumption. In [54] we show that the corresponding cocycles are cohomologically trivial.

Remark 7.8. Leidwanger and Morier-Genoux introduced in [34] a *Jordan superalgebra* in the geometric setting described here. They put

$$\mathcal{J} := \mathcal{A}\oplus\mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}}\oplus\mathcal{J}_{\bar{1}}. \tag{7.14}$$

and define the (Jordan) product $\circ$ via the algebra structures for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f\circ g &:= f\cdot g &&\in\mathcal{A},\\ f\circ\varphi &:= f\cdot\varphi &&\in\mathcal{F}^{-1/2},\\ \varphi\circ\psi &:= [\varphi,\psi] &&\in\mathcal{F}^0.\end{aligned} \tag{7.15}$$

It is a non-associative extension of the associative algebra A of meromorphic functions. By rescaling the second definition with the factor $1/2$ one obtains a *Lie anti-algebra* as introduced by Ovsienko [41].

Of course it is easy again to express everything in terms of our introduced Krichever–Novikov type basis and in particular to calculate the structure equations in the genus zero case completely in the same manner as above. We calculate

$$\begin{gathered} A_n \circ A_m = A_{n+m}, \quad A_n \circ B_m = B_{n+m}, \quad B_n \circ B_m = A_{n+m} + 4A_{n+m+1}, \\ A_n \circ \varphi_m = \varphi_{n+m}, \quad A_n \circ \psi_m = \psi_{n+m}, \quad B_n \circ \varphi_m = \psi_{n+m}, \quad B_n \circ \psi_m = \varphi_{n+m} + 4\varphi_{n+m}, \end{gathered} \tag{7.16}$$

and using (3.14)

$$\begin{aligned} \varphi_n \circ \varphi_m &= 1/2(m-n)B_{n+m}, \\ \varphi_n \circ \psi_m &= 1/2(m-n)A_{n+m} + (2(m-n)+1)A_{n+m+1} = -\psi_m \circ \varphi_n, \\ \psi_n \circ \psi_m &= 1/2(m-n)B_{n+m} + 2(m-n)B_{n+m+1}\,. \end{aligned} \tag{7.17}$$

See also [34]. The almost-graded structure is obvious.

8. Remarks on representations

Having recognized the genus zero algebras as examples of multi-point algebras of KN type, complete collection of their representations as presented e.g. in [55] can be studied. In this article we will not carry this out, but just name them with a few hints.

We start from the natural action of our vector fields, functions, (and more general current Lie algebras) on forms of weight λ. These representations are not of the type one is looking for, e.g. in physics. They do not have a ground state (a vacuum), no creation operators, no annihilation operators. But after choosing a splitting with an induced almost-grading and adapted basis one can construct semi-infinite wedge forms of weight λ. They supply candidates for such desired representations. To extend the natural representation to the wedge-forms we have to regularize the action. The resulting action will only be a projective Lie action. The cocycle defining it defines a central extensions of the algebras under consideration. It turns out that the cocycle is local and as a consequence the central extension is almost-graded. Of course the reference is always the almost-grading induced by the splitting from which we started. We could e.g. start with the standard splitting and obtain exactly the one-dimensional central extensions identified in this article as allowing the extension of the almost-grading, i.e. the ones obtained by taking the residue at ∞.

For $N > 2$ we have more than one splitting and hence more than one almost-grading. For each splitting we will obtain another representation and extension of the original algebra.

Via this process we get semi-infinite wedge representations, or equivalently fermionic Fock space representations of a centrally extended vector field algebra, differential operator algebra, respectively affine Lie algebra [55, Sect. 7]. Furthermore we have $b-c$-systems, and fields in CFT [55, Sect. 8].

As far as the affine Lie algebras are concerned, Verma modules and highest weight representations are given [55, Sect. 9.9]. Again without fixing an almost-grading we cannot even talk about highest weight representations. Of special relevance is the Sugawara representation (energy-momentum representation). Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$ or the Heisenberg algebra. In both cases the defining cocycle for the central extension should be local. Given an "admissible representation", e.g. a highest weight representation, the arbitrary genus, multi-point Sugawara construction done by the author together with Sheinman [56] can applied (see [55, Sect. 10] for some simplifications). The level c of the representation is the scalar by which the central element of the affine algebra acts. Let κ be the dual Coxeter number, respectively $\kappa = 0$ for the Heisenberg algebra. If the level is non-critical, e.g. if $c + \kappa \neq 0$ then the Sugawara operators can be rescaled and the rescaled operators yield a representation of the centrally extended vector field algebra given by a cocycle which is local with respect to the almost-grading we started with. In fact, via the Sugawara operator we obtain a projective representation of $\mathcal{D}^1_{\mathfrak{g}}$ which is the semi-direct product of $\mathfrak{g}$ with $\mathcal{L}$ discussed in Section 3.8. By passing to a central extension we get a honest Lie representation of $\widehat{\mathcal{D}^1_{\mathfrak{g}}}$, see [55, Prop. 10.15]. The Sugawara representations appear in the context of WZNW models, see e.g. [57], [58], [55, Sect. 11], [59] for arbitrary genus. They are also of relevance in genus zero. Some steps in directions of vertex operator algebras are done by Linde [35], [36].

The choice of an almost-grading is an overarching and necessary concept in the theory of representations of our algebras. Note that in genus zero and two points it is always implicitly given by the standard grading (and it is a grading) which allows the representation theory to get of the grounds. For the general situation the choice of an almost-grading does the job. Such an almost-grading always exists. But we have to take into account that we will have finitely many choices as we will have a finite number of different splitting and hence almost-gradings for the same algebra. From the point of the non-extended algebra they correspond to different projective representations.

Another point which can be addressed is the symmetry aspect. In the genus zero situation for $N = 3$ we have additional geometric symmetries which induces automorphism of the algebras (even after fixing the three points). For the generic $N = 4$ case there are no such additional symmetries. But for special choices of these points there might be some inducing also automorphism of the algebras. Klein's list of possible finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ given by C_N, D_N, A_4, A_5, S_4 show up. See [37], [38], [38], [12]. For an approach via KN objects, see [10].

Appendix A. Some useful formulas for the three-point case

A.1. **Definitions.** Our points where poles are allowed are normalized to be $z = 0$, $z = 1$ and $z = \infty$. We define the following basic functions admitting only poles there.

$$\begin{aligned} A_n(z) &:= z^n(z-1)^n, \\ B_n(z) &:= z^n(z-1)^n(2z-1) = A_n(z)\cdot(2z-1). \end{aligned} \tag{A.1}$$

Note that $A_0 = 1$ and $B_0 = 2z - 1$, and that A_n and B_n are holomorphic outside of ∞ if and only if $n \geq 0$. More precisely,

Lemma A.1. *The divisors corresponding to A_n and B_n are*

$$\begin{aligned} (A_n) &= n\cdot[0] + n\cdot[1] - 2n\cdot[\infty], \\ (B_n) &= n\cdot[0] + n\cdot[1] + 1\cdot[1/2] - (2n+1)\cdot[\infty]. \end{aligned}$$

Proof. For finite z values this is obvious. For the order at the point ∞ we have to replace z by $1/w$ with w the local variable at ∞. For example we obtain

$$A_n(z(w)) = w^{-n}(1/w-1)^n = w^{-2n}(1-w)^n. \tag{A.2}$$

Hence, the statement. Accordingly we get the result for B_n. □

The following is obvious.

Lemma A.2. *We have the symmetry*

$$A_n(1-z) = A_n(z), \qquad B_n(1-z) = -B_n(z)\ .$$

A.2. **Products.**

Lemma A.3.

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned}$$

Proof. The first two relations are by the definitions of A_n and B_n. The last follows from

$$(2z-1)^2 = 1 + 4z(z-1). \tag{A.3}$$

□

A.3. **Derivatives.** For the Lie product and for the Lie algebra cocycles we will need the derivatives of our basis functions. They will be linear combinations of the A_n and B_n. We will need their explicit expressions.

Lemma A.4. *For the derivatives of our basic functions we have*

$$\begin{aligned} A_n' &= n\,B_{n-1}, \\ B_n' &= 2(2n+1)\,A_n + n\,A_{n-1}, \\ A_n'' &= 2n(2n-1)\,A_{n-1} + n(n-1)\,A_{n-2}, \\ B_n'' &= 2n(2n+1)\,B_{n-1} + n(n-1)\,B_{n-2}, \\ A_n''' &= 2n(2n-1)(n-1)\,B_{n-2} + n(n-1)(n-2)\,B_{n-3}, \\ B_n''' &= 4n(2n+1)(2n-1)\,A_{n-1} \\ &\qquad + 4n(n-1)(2n-1)\,A_{n-2} + n(n-1)(n-2)\,A_{n-3}\,. \end{aligned}$$

Proof. Starting from $A_n(z) = z^n(z-1)^n$ we obtain

$$A_n'(z) = n\,z^{n-1}(z-1)^{n-1}(2z-1) = n\,B_{n-1}(z).$$

Furthermore,

$$\begin{aligned} B_n' = (A_n \cdot (2z-1))' &= nB_{n-1} \cdot (2z-1) + A_n \cdot 2 \\ &= n\,B_{n-1} \cdot B_0 + 2A_n = 2(2n+1)A_n + nA_{n-1}. \end{aligned}$$

For this we used Lemma A.3. These are the basic results. Now we use them to calculate the higher order derivative. For example $A_n'' = nB_{n-1}'$ and by substituting B_{n-1}' we get the expression in the lemma. By the same way all other results can be calculated. □

Using Lemma A.4 and Lemma A.3 we immediately verify the following relations which will be needed in the main text.

Lemma A.5.

$$\begin{aligned} A_n A_m' &= m\,B_{n+m-1}, \\ A_n B_m' &= 2(2m+1)\,A_{n+m} + m\,A_{n+m-1}, \\ B_n A_m' &= 4m\,A_{n+m} + m\,A_{n+m-1}, \\ B_n B_m' &= 2(2m+1)\,B_{n+m} + m\,B_{n+m-1}. \end{aligned}$$

Lemma A.6.

$$\begin{aligned}
A_nA''_m &= 2m(2m-1)\,A_{n+m-1} + m(m-1)\,A_{n+m-2},\\
A_nB''_m &= 2m(2m+1)\,B_{n+m-1} + m(m-1)\,B_{n+m-2},\\
B_nA''_m &= 2m(2m-1)\,B_{n+m-1} + m(m-1)\,B_{n+m-2},\\
B_nB''_m &= 8m(2m+1)\,A_{n+m} + 2m(4m-1)\,A_{n+m-1} + m(m-1)\,A_{n+m-2}.
\end{aligned}$$

Lemma A.7.

$$\begin{aligned}
A_nA'''_m &= 2m(2m-1)(m-1)\,B_{n+m-2} + m(m-1)(m-2)\,B_{n+m-3},\\
A_nB'''_m &= 4m(2m+1)(2m-1)\,A_{n+m-1} + 4m(m-1)(2m-1)\,A_{n+m-2}\\
&\qquad + m(m-1)(m-2)\,A_{n+m-3},\\
B_nA'''_m &= 8m(2m-1)(m-1)\,A_{n+m-1} + 2m(m-1)(4m-5)\,A_{n+m-2}\\
&\qquad + m(m-1)(m-2)\,A_{n+m-3},\\
B_nB'''_m &= 4m(2m+1)(2m-1)\,B_{n+m-1} + 4m(2m-1)(m-1)\,B_{n+m-2}\\
&\qquad + m(m-1)(m-2)\,B_{n+m-3}.
\end{aligned}$$

A.4. **Residues.** Next we calculate the residues of $A_n dz$ and $B_n dz$ at the points where poles might be. Recall that these are the points $0, 1$ and ∞.

For this aim we need the Laurent series expansion of $(z-1)^m$ around zero. We collect the following well-known facts about binomial series.

The expansion

$$(z-1)^m = \sum_{k=0}^{\infty} \binom{m}{k} z^k (-1)^{m-k}, \qquad z \in \mathbb{C},\ |z| < 1 \tag{A.4}$$

is valid for all $m \in \mathbb{Z}$.

For negative exponents an equivalent expression is

$$(z-1)^{-n} = (-1)^n \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^k, \qquad z \in \mathbb{C},\ |z| < 1, \tag{A.5}$$

where $n \in \mathbb{N}$. Later we will need the easy relation

$$\binom{2k}{k} = \frac{(2k-1)!!}{k!}\, 2^k, \qquad k \in \mathbb{N}, \tag{A.6}$$

where $(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$ is the double factorial.

Lemma A.8. *For the residues at the point* 0 *we have*

$$\operatorname{res}_0(A_{-n}dz) = \begin{cases} 0, & n \le 0,\\ -1, & n = 1,\\ (-1)^n \dfrac{(2n-3)!!}{(n-1)!}\, 2^{n-1}, & n \ge -2. \end{cases}$$

Proof. If $n < 0$ then there is no pole at $z = 0$. Hence let $n > 0$. We use (A.5) and calculate

$$A_{-n}(z) = (-1)^n \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^{k-n}. \tag{A.7}$$

The residue at $z = 0$ is given by the coefficient paired with z^{-1}. Hence it is given by the coefficient for $k = n - 1$

$$\operatorname{res}_0(A_{-n}dz) = (-1)^n \binom{2(n-1)}{n-1}. \tag{A.8}$$

For $n = 1$ we obtain the value -1. For $n > 1$ we use (A.6) and obtain

$$\operatorname{res}_0(A_{-n}dz) = (-1)^n \frac{(2n-3)!!}{(n-1)!}\, 2^{n-1}. \tag{A.9}$$

This was the claim. □

Lemma A.9.

$$\operatorname{res}_0(B_m dz) = \begin{cases} 1, & m = -1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. From Lemma A.4 we conclude $B_m = (1/(m+1))A_{m+1}$ if $m \neq -1$. Hence, for $m \neq -1$ the differential $B_m dz$ is an exact differential and hence does not have any residue. It remains $B_{-1}(z) = z^{-1}(z-1)^{-1}(2z-1)$ which obviously has as residue $+1$ at $z = 0$. □

Lemma A.10.

$$\operatorname{res}_1(A_m dz) = -\operatorname{res}_0(A_m dz)$$
$$\operatorname{res}_1(B_m dz) = \operatorname{res}_0(B_m dz)\ .$$

Proof. We make a change of local coordinates $v = 1 - z$. In particular we have $dv = -dz$. Hence, for the local representing function $\tilde{A}_m(v)$ we have

$$\tilde{A}_m(v) = -A_m(z(v)) = -A_m(1-v) = -A_m(v),$$

and $\tilde{B}_m(v) = B_m(v)$. Moreover, the point $z = 1$ corresponds to the point $v = 0$. As the residue is independent of the choice of local coordinates we obtain exactly the statement of the lemma. □

Lemma A.11.

$$\begin{aligned} \operatorname{res}_\infty(A_m dz) &= 0, \\ \operatorname{res}_\infty(B_m dz) &= -2\operatorname{res}_0(B_m dz), \\ &= \begin{cases} -2, & m = -1, \\ 0, & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. By the residue theorem [52] for a compact Riemann surface the sum over all residues of a meromorphic differential is zero. As our differentials have only poles at $0, 1, \infty$ the claim follows from Lemmas A.8, A.9, and A.10. □

Remark A.12. The proofs of Lemmas A.8 and A.9 do not really make reference to the $N = 3$ situation. But for the proofs of the Lemmas A.10 and A.11 the symmetry for $N = 3$ is used. Nevertheless, the principal statement in Lemma A.11, telling us that only for finitely many m there might be a non-vanishing residue, remains true. To indicate this I present an alternative proof of Lemma A.11 which generalizes to arbitrary N.

Proof. We make the change of coordinates

$$z = \frac{1}{w}, \qquad dz = -\frac{1}{w^2}dw \tag{A.10}$$

and express the elements $A_m(z)dz$ and $B_m(z)dz$ in the new coordinates. In the first case we obtain

$$-w^{-2m-2}(1-w)^m dw, \tag{A.11}$$

in the second case

$$-w^{-2m-3}(1-w)^m(2-w)dw. \tag{A.12}$$

For the existence of a pole at ∞ it is necessary that $-2m-2<0$ respectively $-2m-3<0$. Hence, $m>-1$, respectively $m\ge -1$. If $m\ge 0$ there are no poles at the other points, hence by the residue theorem there is no residue at ∞. This says that in the first case there is no residue possible at all and in the second case only for $m=-1$. It calculates as

$$-\operatorname{res}_\infty(w^{-1}(1-w)^{-1}(2-w)dw)=-2. \tag{A.13}$$

□

Appendix B. Three-point $\mathfrak{sl}(2,\mathbb{C})$-current algebra for genus 0

In this appendix we will give the example of the universal central extension of the 3-point $\mathfrak{sl}(2,\mathbb{C})$-current algebra. The general theory has been developed in Section 5.5. The 3-point $\overline{\mathfrak{sl}}(2,\mathbb{C})$ algebra is of relevance in quite a number of applications, we only name statistical mechanics [23], [24]. Hence, the explicit knowledge of the structure equations with respect to some basis might be of some interest. We take $\mathfrak{sl}(2,C)$ with the standard matrix generators

$$H:=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X:=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y:=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \tag{B.1}$$

and induced relations

$$[H,X]=2X,\quad [H,Y]=-2Y,\quad [X,Y]=2X\ . \tag{B.2}$$

As basis elements for the current algebra $\overline{\mathfrak{sl}}(2,\mathbb{C})$ with respect to the almost-grading introduced in Section 6 we take the elements

$$Z^{(s)}=Z\otimes A_n,\quad Z^{(a)}=Z\otimes B_n,\qquad Z\in\{H,X,Y\}. \tag{B.3}$$

The structure of the current algebra comes via (3.20) from $\mathcal{A}$ (and of course from $\mathfrak{g}$). We need the Cartan-Killing form. Up to a normalization it is given by

$$\beta(A,B)=\operatorname{tr}(A\cdot B). \tag{B.4}$$

Hence,

$$\beta(X,Y)=\beta(Y,X)=1,\quad \beta(H,H)=2,\quad \beta(H,X)=\beta(H,Y)=\beta(X,H)=\beta(Y,H)=0. \tag{B.5}$$

The universal central extension will have a two-dimensional center and will be given by

$$[Z\otimes f,W\otimes g]=[Z,W]\otimes f\cdot g+\alpha_\infty\cdot\beta(Z,W)\cdot\gamma^{\mathcal{A}}_\infty(f,g)\cdot t_\infty+\alpha_0\cdot\beta(Z,W)\cdot\gamma^{\mathcal{A}}_0(f,g)\cdot t_0, \tag{B.6}$$

with t_∞,t_0 central elements, $\alpha_\infty,\alpha_0\in\mathbb{C}$. Recall that $\gamma^{\mathcal{A}}_a(f,g)$ can be calculated as $\operatorname{res}_a(fdg)$.

From the general theory developed in Section 5.5 we know that the central extension will be almost-graded with respect to the standard splitting if and only if $\alpha_0=0$.

All the data needed has been calculated already before. If we collect them we obtain the following results.

$$\begin{aligned}
&[H^{(s)}_n,H^{(s)}_m]=\alpha_\infty\cdot 4n\cdot\delta_m^{-n}\cdot t_\infty-\alpha_0\cdot 2n\cdot\delta_m^{-n}\cdot t_0,\\
&[H^{(s)}_n,H^{(a)}_m]=2\alpha_0\left(n\,\delta_m^{-n}+2n\,\delta_m^{-n-1}+\sum_{k=2}^{\infty}n\,(-1)^{k-1}2^k\frac{(2k-3)!!}{k!}\delta_m^{-n-k}\right)\cdot t_0.\\
&[H^{(a)}_n,H^{(a)}_m]=\alpha_\infty(4n\delta_m^{-n}+8(2n+1)\delta_m^{-n-1})\cdot t_\infty-\alpha_0(2n\delta_m^{-n}+4(2n+1)\delta_m^{-n-1})\cdot t_0\\
&[H^{(s)}_n,X^{(s)}_m]=2X^{(s)}_{n+m},\quad [H^{(s)}_n,X^{(a)}_m]=2X^{(a)}_{n+m},\quad [H^{(a)}_n,X^{(a)}_m]=2X^{(s)}_{n+m}+8X^{(s)}_{n+m},\\
&[H^{(s)}_n,Y^{(s)}_m]=-2Y^{(s)}_{n+m},\quad [H^{(s)}_n,Y^{(a)}_m]=-2Y^{(a)}_{n+m},\quad [H^{(a)}_n,Y^{(a)}_m]=-2Y^{(s)}_{n+m}-8Y^{(s)}_{n+m},
\end{aligned}$$

$$
\begin{aligned}
[X_n^{(s)}, Y_m^{(s)}] &= H_{n+m}^{(s)} + \alpha_\infty \cdot 2n \cdot \delta_m^{-n} \cdot t_\infty - \alpha_0 \cdot n \cdot \delta_m^{-n} \cdot t_0, \\
[X_n^{(s)}, Y_m^{(a)}] &= H_{n+m}^{(a)} + \alpha_0 \left(n\, \delta_m^{-n} + 2n\, \delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1} 2^k \frac{(2k-3)!!}{k!} \delta_m^{-n-k} \right) \cdot t_0. \\
[X_n^{(a)}, Y_m^{(a)}] &= H_{n+m}^{(s)} + 4H_{n+m+1}^{(s)} + \alpha_\infty (2n\delta_m^{-n} + 4(2n+1)\delta_m^{-n-1}) \cdot t_\infty \\
&\quad - \alpha_0 (n\delta_m^{-n} + 2(2n+1)\delta_m^{-n-1}) \cdot t_0 \,.
\end{aligned}
$$

Of course, the elements t_∞ and t_0 are central and we have anti-symmetry. The local cocycle, i.e. the cocycle coming with t_∞ was given in [53] and reproduced in [55, Equ. 12.75]. Unfortunately, there the central terms related to $[H_n^{(.)}, H_m^{(.)}]$ were forgotten.

Remark B.1. By Cox and Jurisich [13, Thm.2.4] a different form of a universal central extension for the $\mathfrak{sl}(2,\mathbb{C})$ current algebra was proposed. This form was taken up in [14]. An inspection of the structure equation shows that in the proposed form two independent cocycles coming with the central elements ω_0 and ω_1 show up. Both would be local. But this contradicts the uniqueness of the local cocycle class (up to rescaling) as obtained in [51], which was also recalled in the current article. A closer examination shows that if "$\omega_1 \neq 0$" the proposed structure constants do not define a Lie algebra. The reader might check himself, that e.g. the Jacobi identity of the triple $(f^1_{-(n+m+2)}, h^1_m, e^1_n)$ (in the notation of the quoted articles) will be a non-zero multiple of ω_1.

The principal structure, as far as the central extension is concerned, in particular also that we have one unique local cocycle class (up to rescaling) does not depend in an essential manner on the simple Lie algebra. See also Bremner [7] for the example of the 4-point case. Here the universal central extension is 3-dimensional, One of the classes will be local with respect to the standard splitting, the other two not. The latter two are "coupled" with ultrasperical (Gegenbauer) polynomials. I like also to mention that in [3] also the 3-point case was considered in another basis exhibiting another symmetry useful in the context of statistical mechanics.

Remark B.2. As an additional example we like to give the case of the current algebra of $\mathfrak{gl}(n,\mathbb{C})$ for the N-point case. Of course, as $\mathfrak{gl}(n,\mathbb{C})$ is not perfect it does not admit a universal central extension. But by the classification results we can give the maximal central extension for which the cocycles are multiplicative (or $\mathcal{L}$-invariant)

$$
\begin{aligned}
[x \otimes f, y \otimes g] &= [x,y] \otimes f \cdot g + \sum_{i=1}^{N-1} \alpha_i \cdot \mathrm{tr}(x \cdot y)\, \mathrm{res}_{a_i}(f dg) \cdot t_i \\
&\quad + \sum_{i=1}^{N-1} \beta_i \cdot \mathrm{tr}(x)\mathrm{tr}(y)\, \mathrm{res}_{a_i}(f dg) \cdot s_i \,.
\end{aligned}
\tag{B.7}
$$

Here $\alpha_i, \beta_i \in \mathbb{C}$ and t_i and s_i are central. In this context see [55, Sect. 9.8].

Appendix C. Projective and affine connections

Let $(U_\alpha, z_\alpha)_{\alpha \in J}$ be a covering of the Riemann surface Σ_g by holomorphic coordinates with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$.

Definition C.1. (a) A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *projective connection* if it transforms as

$$
R_\beta(z_\beta) \cdot (f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{C.1}
$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

(b) A system of local (holomorphic, meromorphic) functions $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *affine connection* if it transforms as

$$T_\beta(z_\beta) \cdot (f'_{\beta,\alpha}) = T_\alpha(z_\alpha) + \frac{f''_{\beta,\alpha}}{f'_{\beta,\alpha}}. \tag{C.2}$$

Every Riemann surface admits a holomorphic projective connection [22],[20]. Given a point P then there exists always a meromorphic affine connection holomorphic outside of P and having maximally a pole of order one there [47].

From their very definition it follows that the difference of two affine (projective) connections will be a (quadratic) differential. Hence, after fixing one affine (projective) connection all others are obtained by adding (quadratic) differentials.

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